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Stochastic stability of dynamic user equilibrium in unidirectional networks: Weakly acyclic game approach

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Abstract

The aim of this study is to analyze the stability of the dynamic user equilibrium (DUE) with fixed departure times in unidirectional networks. Specifically, the stochastic stability of the equilibrium, which is the concept of stability in a day-to-day dynamics subjected to stochastic effects, is herein considered. To achieve this, a new approach is developed by synthesizing the three concepts: the decomposition technique of DUE assignments, the weakly acyclic game, and the asymptotic analysis of the stationary distribution of the dynamics. Specifically, we first formulate the DUE assignment as a strategic game (DUE game), which deals with atomic users. We then prove that there exists an appropriate order of assigning users one by one to the network for ensuring an equilibrium. With this property, we prove that the DUE game is a weakly acyclic game, which is a generalization of potential games. The convergence and stochastic stability of the DUE game are then established based on the theory of weakly acyclic games. Finally, numerical experiments are conducted to validate these theoretical results.

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Keywords: dynamic user equilibrium; Nash equilibrium; unidirectional network; weakly acyclic games; convergence; stochastic stability

1. Introduction

There remain several issues regarding the theoretical properties of the dynamic user equilibrium (DUE) (Iryo, 2013), such as existence, uniqueness, and stability. Stability is a particularly important property for equilibrium states to be realized in a transportation system. If the stability of an equilibrium is not guaranteed, then the equilibrium cannot be preserved against small perturbations. As a result, the corresponding equilibrium flow pattern would just be an extreme state of rare occurrence (Beckmann et al., 1956).

In the analysis of the stability of an equilibrium, we first consider a day-to-day dynamics that describes how users change their travel choices according to the current traffic state, such as the Smith dynamics (Smith, 1984),

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projection dynamics (Zhang and Nagurney, 1996), and other evolutionary game dynamics (see Sandholm, 2010). We then investigate the local and/or global behaviors of the dynamics[1]. In particular, a powerful tool for the latter is a Lyapunov function for the dynamics. For example, Smith (1984) showed the convergence of the Smith dynamics to a set of equilibria in a static traffic assignment when the route travel time functions are monotonic. Moreover, if the functions are strictly monotonic, the unique equilibrium becomes convergent and stable[2] (i.e., asymptotically stable). For a DUE problem with fixed departure times, Smith and Ghali (1990) showed the monotonicity of the route travel time functions in a network in which each route contains only one bottleneck (called single-bottleneck-per-route network). Utilizing this property, Mounce (2006) proved the existence of the Lyapunov function for the Smith dynamics and demonstrated the convergence of the dynamics to a set of equilibria in the network.

While the monotonicity is a desirable property for proving the existence of a Lyapunov function, it is not a sufficient condition for a DUE to be stable because the existence of the Lyapunov function guarantees the convergence to a set of equilibria only. This means that small perturbations might lead the state away from an equilibrium point to another equilibrium point. Moreover, the monotonicity is not a general property of DUE problems. It is known that the monotonicity is not guaranteed, even in simple cases, such as when a route has two or more bottlenecks (Kuwahara, 1990; Mounce and Smith, 2007). Thus, it may be difficult to apply the Lyapunov approach to the DUE problems in the networks other than the single-bottleneck-per-route networks; we need to develop a different approach to examine the stability of a DUE in the networks in which the route travel time function are not monotonic.

In this study, we analyze the stability of a DUE with fixed departure times through a novel analytical approach synthesizing several concepts, each of which is developed in dynamic traffic assignment or in game theory. This approach consists of the following three concepts: (A) the decomposition technique of DUE assignments, (B) the weakly acyclic game, which is a class of strategic games, and (C) the asymptotic analysis of stationary distribution of a day-to-day dynamics subject to perturbations (perturbed dynamics). More specifically, we study DUE problems in unidirectional networks (Iryo and Smith, 2017)[3] to which the concept (A) can apply as we will prove in this study. In the analysis, we first formulate a DUE assignment as a strategic game (we call it "DUE game"), which deals with atomic users. We then prove the existence of an appropriate order of assigning users one by one to the network for ensuring equilibrium. With this ordering property, we further establish the relationship between the DUE game and the weakly acyclic game. We finally examine the stochastic stability of equilibrium (Young, 1993), which is the stability concept in a perturbed dynamics, based on the theory of weakly acyclic game. Below, we outline our approach with brief reviews of each concept utilized in this study.

(A) The decomposition technique of DUE assignments, which is proposed by Kuwahara and Akamatsu (1993), is a concept for analyzing an equilibrium through the decomposition of a DUE assignment and is applicable to the special class of networks, such as single-origin or single-destination networks. For example, in a single-origin network, the DUE assignment can be decomposed with respect to the departure time from the unique origin, i.e., we can solve the DUE assignment sequentially in the order of the departure time. This is because the equilibrium concept along with the first-in-first-out (FIFO) principle and the causality implies that users departing from the origin earlier must arrive at any node not later than the others leaving the same origin later. In other words, the order of departure from the origin must be kept at any intermediate node.

This technique has been utilized to investigate the properties of equilibrium analytically (Akamatsu, 2000; Akamatsu et al., 2015; Wada et al., 2018) and to develop solution algorithms (Akamatsu, 2001; Waller and Ziliaskopoulos, 2006) for the model with continuum users. On the other hand, Iryo (2011) and Satsukawa and Wada (2017) explored the applicability of the technique to the DUE game by using a slightly generalized ordering property than one for the continuum model, which is formally introduced later. As we will show in this paper, the DUE game with this property can be viewed as the weakly acyclic game that is the basis of our analysis of the stochastic stability.

[1] Since the stability is examined subject to the applied dynamics, its property can be different under other dynamics.

[2] We explicitly distinguish convergence from stability based on the definitions in previous studies (e.g., Watling, 1998; Watling and Cantarella, 2013), although the former is sometimes called stability (e.g., Smith, 1984). Specifically, we define an equilibrium as stable if we can ensure that the state is arbitrarily close to the equilibrium for all times, and as convergent if the state will approach the equilibrium as time approaches infinity.

[3] The class of unidirectional networks is more general than the class of single-bottleneck-per-route networks in the sense that it includes multiple-bottleneck-per-route networks. However, there does not exist the simple inclusion relation between two classes.

(B) The weakly acyclic game (Young, 1993; Marden et al., 2009) is a class of strategic games in which there exist better response paths (i.e., sequences of users' better response) from every state to a (pure) Nash equilibrium. In this game, a day-to-day dynamics always converges to a set of equilibria under a mild condition on the dynamics. These rest points of the dynamics play a central role in the analysis of the existence of a stochastically stable equilibrium.

(C) The analysis of a stationary distribution is useful for investigating the dynamical behavior of a transportation system subject to stochastic effects. In this analysis, we first consider the perturbed dynamics, which is continually perturbed by time-independent small mutations or mistakes. This dynamics is a Markov chain on a finite state space and has a unique stationary distribution. Thus, we can investigate the long run behavior of the dynamical system regardless of the initial states. In the traffic assignment field, several studies investigated the stationary distribution of link/route flows, i.e., mean and expected values of flows (e.g., Cascetta, 1989; Cantarella and Cascetta, 1995; Watling, 1998; Balijepalli and Watling, 2005; Cantarella and Watling, 2016).

Meanwhile, in game theory, the property on stability of an equilibrium point has been investigated from the asymptotic behavior of the stationary distribution. Specifically, it has been shown that, when the perturbation becomes small, the distribution becomes concentrated around particular states (Foster and Young, 1990; Young, 1993). These are called stochastically stable states, which will be observed frequently in the long run when the perturbation is small. It can be regarded as a plausible equilibrium to be realized in the real world. These stable states are contained in a set of the rest points of the unperturbed version of the dynamics. This means that, in a weakly acyclic game, there always exists at least a stochastically stable equilibrium when the convergence of the unperturbed dynamics is guaranteed.

As a result, we can establish a novel result on the stability of DUE by synthesizing these concepts. While each of the concepts is developed in each research field, this study presents several unique contributions. First, we establish the stability and convergence results of DUE in a unidirectional network whereas previous studies analyzing these properties have been limited to the single-bottleneck-per-route network. Second, we find that strict improvement of user's utility in a dynamics is the important condition for ensuring the existence of a stochastic stability of the DUE. Third, we demonstrate the new role of the decomposition technique in the analysis of the theoretical properties of DUE, such as convergence, stochastic stability, and establishing the relationship with game theory.

The remainder of this paper is organized as follows. In Section 2, we introduce the definition of the DUE game and a decomposition-based solution algorithm. Section 3 presents the proof of the ordering property of a DUE game in a unidirectional network with the decomposition technique. Section 4 establishes the relationship between the DUE game and weakly acyclic game. Based on this relationship, we investigate the convergence and stochastic stability of equilibrium in this section and in Section 5. Section 6 presents the numerical experiments conducted to demonstrate the theoretical properties. Section 7 concludes the paper.

2. DUE game and decomposition solution algorithm

In this section, we first define a strategic game of a DUE assignment that deals with atomic users, which we call "DUE game". The DUE game consists of a road network, vehicles traveling through the network (atomic users), action sets (route choice sets), and utilities of vehicles (travel times of users). The Nash equilibrium of this game is considered as the user equilibrium of dynamic traffic assignments. Then, we describe a decomposition-based solution algorithm for the DUE game.

2.1. Definition

2.1.1. Network

A general road network with many-to-many origin-destination (OD) demands is herein considered. The network consists of a set of nodes $\mathcal{N}$ and a set of directed links $\mathcal{L}$. Sets of origin nodes and destination nodes are denoted by $\mathcal{N}_o$ and $\mathcal{N}_d$, respectively. A set of all acyclic routes from node a to node b is denoted by $\mathcal{R}(a, b)$. When these nodes are not connected, $\mathcal{R}(a, b) = \emptyset$. $\mathcal{N}(r)$ is a set of nodes included in route r.

2.1.2. Users and strategies

Each vehicle is considered as an atomic user of the game. A set of users is denoted by $\mathcal{P}$ and the number of the users is denoted by $|\mathcal{P}|$. The origin, destination, and departure time of user $i \in \mathcal{P}$ are denoted by o_i, d_i, and s_i, respectively. These are given exogenously. All users departing from the same origin have different departure times.

User i can choose any acyclic route r_i to travel between the user's origin and destination. A set of routes of user i is denoted by $\mathcal{R}_i$ $(= \mathcal{R}(o_i, d_i))$. This set includes the option ϕ_i when user i does not select any route (i.e., user i is not assigned onto the network). The users selecting ϕ_i are called *non-assigned users*. We utilize this class of users in a decomposition-based solution algorithm. A route choice profile of all users is denoted by vector $\mathbf{r} \equiv \{r_1, \ldots, r_i, \ldots, r_{|\mathcal{P}|}\} \in \mathcal{R}$ where $\mathcal{R} \equiv \mathcal{R}_1 \times \cdots \times \mathcal{R}_{|\mathcal{P}|}$. In addition, the route choice profile of users other than user i is denoted by $\mathbf{r}_{-i} \equiv \{r_1, \ldots, r_{i-1}, r_{i+1}, \ldots, r_{|\mathcal{P}|}\}$. With this notation, we sometimes represent a profile $\mathbf{r}$ as $(r_i, \mathbf{r}_{-i})$ when a route choice of a certain user should be clearly stated.

2.1.3. Utility and Nash equilibrium

A disutility of each user is equal to the travel time of the user. Because the departure time of each user is fixed, the user's arrival time at the destination can be regarded as the user's disutility. In this regard, $g_i(r_i; \mathbf{r}_{-i})$ denotes the arrival time of user i when the route choice of the user is r_i and the route choice profile of the other users is $\mathbf{r}_{-i}$. Note that we set the arrival time of a non-assigned user to infinity: $g_i(\phi_i; \mathbf{r}_{-i}) = \infty$. We denote by $u_n(o_i, s_i, r_i; \mathbf{r}_{-i})$ the arrival time at node n when a user departing from $o \in \mathcal{N}_o$ at time s_i takes route r_i and the other users' route choices are $\mathbf{r}_{-i}$.

The node arrival times are determined by a dynamic loading model when the route choices of all users are determined. Any appropriate model which satisfies a few natural conditions for dynamic traffic assignment can be employed. Note that the FIFO principle and causality (Carey et al., 2003) must hold. A physical queue concept (e.g., the car-following model by Newell, 2002) can be incorporated into the model.

Under the above settings, a DUE is defined as a pure Nash equilibrium of the game. In the pure Nash equilibrium, all users choose their routes to minimize their travel times. This corresponds to the situation where all users choose the best response strategy $\mathbf{r}^*$, which is mathematically described as

$$g_i(r_i^*; \mathbf{r}_{-i}^*) \leq g_i(r_i; \mathbf{r}_{-i}^*), \quad \forall r_i \in \mathcal{R}_i, \forall i \in \mathcal{P}. \tag{1}$$

Furthermore, if Eq. (1) holds with strict inequality for all users, then $\mathbf{r}^*$ is called a *strict* Nash equilibrium.

2.2. Decomposition-based solution algorithm for atomic users

Iryo (2011) proposed a *decomposition-based solution algorithm*, which obtains the Nash equilibrium of a DUE game dealing with atomic users by assigning users one by one to a network in an appropriate order. Although such an order for ensuring equilibrium is not expected in general networks, we will prove its existence in unidirectional networks in Section 3.

In this algorithm, we first consider the situation where all users are non-assigned users, i.e., we set the initial route choice of each user $i \in \mathcal{P}$ to ϕ_i. A set of non-assigned users for the route choice profile $\mathbf{r}$ is denoted by $\overline{\mathcal{P}}(\mathbf{r})$. Then, the algorithm assigns each non-assigned user one by one onto the user's shortest route in an order such that a user who is assigned to the network will not be overtaken by users who will be assigned later. This assignment order is not necessarily equal to the order of departure times of users because a user who has a later departure time may overtake users who have earlier departure times with different origins[4]. The equilibrium is obtained when all users are assigned to the network (i.e., $r_i \neq \phi_i, \forall i \in \mathcal{P}$).

To determine the order mentioned above, this algorithm utilizes the concept of *earliest non-assigned user*. An earliest non-assigned user is the non-assigned user that is not overtaken by other non-assigned users when the user takes the shortest route. For the formal definition of the earliest non-assigned user, we define the earliest arrival time at a node and the shortest route of users as follows:

Definition 1. (**Earliest arrival time at a node**). Consider a route choice profile $\mathbf{r} \in \mathcal{R}$ and user i who has departure time s_i from origin o_i. We denote by $\mathcal{N}_i$ the set of nodes included in at least one route in the route choice set of the user $\mathcal{R}_i$, i.e., $\mathcal{N}_i = \{n \mid n \in \mathcal{N}(r), r \in \mathcal{R}_i\}$. Then, the earliest arrival time of user i at node $n \in \mathcal{N}_i$ for the fixed route choice profile $\mathbf{r}_{-i}$ is denoted by $u_n^*(o_i, s_i; \mathbf{r}_{-i})$, and defined as follows:

$$u_n^*(o_i, s_i; \mathbf{r}_{-i}) = \min_{r \in \{r' | r' \in \mathcal{R}_i, n \in \mathcal{N}(r')\}} u_n(o_i, s_i, r; \mathbf{r}_{-i}). \tag{2}$$

[4] The solution algorithm in which the users are assigned to the network one by one as the order advances must be distinguished from a simulation in which the users enter the network in the order of their departure times and move simultaneously as time advances.

Definition 2. (**Shortest route**). Consider a route choice profile $\mathbf{r} \in \mathcal{R}$. The shortest route of user i is denoted by r_i^* and is defined as follows:

$$r_i^* \in \arg\min_{r \in \mathcal{R}_i} \; g_i(r; \mathbf{r}_{-i}), \tag{3}$$

$$\text{s.t.} \quad u_n(o_i, s_i, r_i^*; \mathbf{r}_{-i}) = u_n^*(o_i, s_i; \mathbf{r}_{-i}), \quad \forall n \in \mathcal{N}(r_i^*). \tag{4}$$

This means that the arrival time at any node included in the shortest route is the same with or earlier than when the user does not take the shortest route, i.e., the dynamic programming (DP) principle holds. Eq. (4) is naturally satisfied in the implementation of the algorithm by employing the DP based algorithm, such as Dijkstra's algorithm. However, it should be explicitly included in the definition because there can exist routes that satisfy Eq. (3) but not satisfy Eq. (4) in theory and such a route is an obstacle to proceed with the analyses in the later sections. A set of the shortest routes of a user departing from origin o at s for the destination d is denoted by $\mathcal{SR}(o, s, d; \mathbf{r})$.

We are now ready to define the earliest non-assigned user as follows:

Definition 3. (**Earliest non-assigned user**). Consider a route choice profile $\mathbf{r} \in \mathcal{R}$ such that $\overline{\mathcal{P}}(\mathbf{r}) \neq \emptyset$. A non-assigned user $i \in \overline{\mathcal{P}}(\mathbf{r})$ is the earliest non-assigned user when the user has the shortest route $\overline{r}_i^*$ satisfying the following condition[5]:

$$u_n(o_i, s_i, \overline{r}_i^*; \mathbf{r}_{-i}) \leq u_n^*(o_j, s_j; \mathbf{r}_{-i}), \quad \forall n \in \mathcal{N}(\overline{r}^*) \cap \mathcal{N}_j, \forall j \in \overline{\mathcal{P}}(\mathbf{r}) \setminus \{i\}. \tag{5}$$

Note that when $|\overline{\mathcal{P}}(\mathbf{r})| = 1$, the non-assigned user automatically becomes the earliest non-assigned user.

When earliest non-assigned user i takes the shortest route $\overline{r}_i^*(\mathbf{r})$ satisfying Eq. (5), the other non-assigned users cannot overtake user i. By combining this property with the FIFO principle and causality of the dynamic loading model, it is guaranteed that the travel time of the shortest route $\overline{r}_i^*(\mathbf{r})$ is independent of the route choices of the other non-assigned users. This implies that, among the non-assigned users, the earliest non-assigned user should have the earliest assignment order for ensuring equilibrium. Therefore, by assigning the earliest non-assigned users to their shortest routes one by one, all users come to choose the ex-post shortest routes, i.e., the Nash equilibrium (Eq. (1)) is obtained. The decomposition-based solution algorithm is now summarized as follows.

Decomposition-based solution algorithm for DUE games (Iryo, 2011)

0. **Initial setting**: Let the number of step $m = 0$ and all users are set as non-assigned users $\overline{\mathcal{P}}(\mathbf{r}^m) = \mathcal{P}$, i.e., $r_i^m = \phi_i, \forall i \in \mathcal{P}$, where r_i^m represents the route choice of user i at step m.
1. **Choose the earliest non-assigned user**: Find the earliest non-assigned user $i \in \overline{\mathcal{P}}(\mathbf{r}^m)$ according to Eq. (5) for a given route choice profile $\mathbf{r}^m$. If there exist multiple earliest non-assigned users, pick one of them[6].
2. **Update the route choice profile and set of non-assigned users**: Assign user i to the shortest route $\overline{r}_i^*(\mathbf{r}^m)$ by using an appropriate dynamic loading model. Let $\mathbf{r}^{m+1} := (\overline{r}_i^*(\mathbf{r}^m), \mathbf{r}_{-i}^m)$ and $\overline{\mathcal{P}}(\mathbf{r}^{m+1}) := \overline{\mathcal{P}}(\mathbf{r}^m) \setminus \{i\}$.
3. **Judge the convergence**: If $\overline{\mathcal{P}}(\mathbf{r}^{m+1}) = \emptyset$, then terminate the algorithm; $\mathbf{r}^{m+1}$ is a Nash equilibrium. Otherwise, let $m := m + 1$, and go back to Step 1.

From its logic, it can be seen that the existence of the appropriate order for ensuring equilibrium corresponds to the existence of the earliest non-assigned user at every step. Thus, in the next section, we prove the existence of the earliest non-assigned user in a unidirectional network.

[5] The node arrival time of a non-assigned user is constrained by assigned users only but not by other non-assigned users because non-assigned users are not loaded to the network yet and not physically interact with each other. Thus, the equality in Eq. (5) may hold.

[6] The resulting equilibrium state is dependent on how to select the earliest non-assigned user among them.

3. DUE game in unidirectional networks

3.1. Unidirectional network and its properties

The unidirectional network is a generalization of single-origin or single-destination networks in the sense that the decomposition technique is applicable to the DUE games. More specifically, for the DUE games in these networks, the earliest arrival times of all nodes from all origins can be represented by functions of a *reference time* of a reference node. We call these functions *node potential functions*. For example, in a single-origin network, the earliest arrival times can be functions of departure time from the origin, as shown in Kuwahara and Akamatsu (1993).

Besides the existence of such a (global) reference time for the DUE problems, there exists a monotonic relation between the reference time and the earliest arrival time of each node. That is, the earliest arrival time of each node does not go back in time if the reference time advances. In a single-origin network, this ordering property is ensured because, as mentioned in Section 1 and Kuwahara and Akamatsu (1993), the equilibrium concept along with the FIFO principle and the causality implies that users departing from the origin earlier must arrive at any node not later than the others leaving the same origin later. We thus can decompose the DUE problems with the ordering property in the order of the reference time.

To define the unidirectional network formally, we first introduce the possible dynamical link travel time profiles according to Iryo and Smith (2017). These profiles are utilized to guarantee the existence of the node potential function independently of the route choices and demand patterns. In other words, a unidirectional network has the node potential function regardless of which patterns of the following possible dynamical link travel time profiles are given:

Definition 4. (**Possible dynamical link travel time profiles**) The possible dynamical link travel time profile on link l is a set of functions defined by

$$C_l^P = \left\{ c_l(t) \mid c_l(t) \in \left\{ \text{Any Lipschitz continuous function} \quad \text{s.t. } c_l(t) \geq c_l^{FT} \text{ and } \frac{c_l(t') - c_l(t)}{t' - t} \geq -1 \right\} \right\}. \tag{6}$$

where $c_l(t)$ is the travel time of link l for a user entering at time t, and c_l^{FT} is the free flow travel time of link l.

This profile represents the feasible set of the link travel time functions with the dynamic loading model satisfying the FIFO principle. The vector of $c_l(t)$ for all links is denoted by $\mathbf{c}$. $C^P = \prod_{l \in \mathcal{L}} C_l^P$ is used to specify the Cartesian product of C_l^P for all links.

Definition 5. (**Unidirectional network**). Given a network, choose an arbitrary origin as a reference node and denote it by $o_{REF} \in \mathcal{N}_o$. Then, the network is the unidirectional network if and only if for any $\mathbf{c} \in C^P$, there exists a potential function $p_n(t; \mathbf{c})$ on node n as a function of a reference time t as follows:

$$p_n(t; \mathbf{c}) = u_n^*(o, p_o(t; \mathbf{c}); \mathbf{c})), \quad \forall n \in \mathcal{N}(r^*), \exists r^* \in \mathcal{SR}(o, p_o(t; \mathbf{c}), d; \mathbf{c})), \forall o \in \mathcal{N}_o . \forall d \in \mathcal{N}_d. \tag{7}$$

$$\text{where} \quad p_{o_{REF}}(t; \mathbf{c}) = t. \tag{8}$$

where $u_n^*(o, p_o(t; \mathbf{c}); \mathbf{c}))$ is the earliest arrival time at node n for given link travel time profiles $\mathbf{c}$.

For a given link travel time profile $\mathbf{c}$, Eq. (7) shows that potential function $p_n(t; \mathbf{c})$ on node n having reference time t is defined as the node earliest arrival time from any origin o at time $p_o(t; \mathbf{c})$ by taking the shortest route r^* to destination d. To describe the potential function for a given route choice profile $\mathbf{r}$, we represent the link travel time profile as a function of the route choice profile $\mathbf{c}(\mathbf{r})$; as a result, the other variables can be defined as functions of $\mathbf{r}$ as follows: $u_n^*(o, s; \mathbf{r}) \equiv u_n^*(o, s; \mathbf{c}(\mathbf{r}))$ and $p_n(t; \mathbf{r}) \equiv p_n(t; \mathbf{c}(\mathbf{r}))$.

We then show the monotonic relation between the node potential and the reference time, which is the extension of Theorem 2 by Iryo and Smith (2017)[7].

[7] While Iryo and Smith (2017) showed a stronger relation, $t < t' \Rightarrow p_n(t; \mathbf{r}) < p_n(t'; \mathbf{r})$, $\forall n \in \mathcal{N}$, but it requires an additional strong condition named the *positive flow assumption*.

Theorem 1. Consider a unidirectional network with the route choice profiles $\mathbf{r}$ and reference times t and t'. Then, the following relationship is obtained:

$$t < t' \Rightarrow p_n(t;\mathbf{r}) \leq p_n(t';\mathbf{r}), \quad \forall n \in \mathcal{N}. \tag{9}$$

Proof. We prove the theorem by combining the following Lemma 1 and 2. We here omit $\mathbf{c}(\mathbf{r})$ in the proofs for simplicity.

We first prove Lemma 1 that states the monotonic relation between the potential on each origin and the reference time.

Lemma 1. Consider a unidirectional network with the route choice profiles $\mathbf{r}$ and reference times t and t'. Then, the following relation holds:

$$t < t' \Rightarrow p_o(t) \leq p_o(t'), \quad \forall o \in \mathcal{N}_o. \tag{10}$$

Proof. The outline of the proof is described as follows. We first establish the relations between potentials on nodes that are included in the shortest routes from the same origin. Specifically, we first consider the shortest routes of a user departing from an arbitrary origin $o \in \mathcal{N}_o$ at the potential $p_o(t)$ and $p_o(t')$. For the potentials on nodes o and n included in both of the shortest routes, we have

$$\begin{cases} p_o(t) \leq p_o(t') \Rightarrow p_n(t) \leq p_n(t'), \\ p_n(t) \leq p_n(t') \Rightarrow p_o(t) \leq p_o(t') \end{cases}. \tag{11}$$

Thus, for a given $t < t'$ on the reference node (origin), we obtain Eq. (10) by recursively applying Eq. (11) to from the reference node to all origins and other nodes (see Appendix A for details). □

Next, we prove Lemma 2 from the definition of unidirectional networks, which states that the potential can be defined on all nodes including those that are not on the shortest routes without violating Eq. (7).

Lemma 2. Consider a unidirectional network with the route choice profiles $\mathbf{r}$. Then, the potential $p_n(t;\mathbf{r})$ on node n having reference time t can be expressed as the minimum value of the earliest arrival times of users departing at the potentials having the same reference time on their origins. That is,

$$p_n(t) = \min_{o \in \mathcal{N}_o} u^*_n(o, p_o(t)), \quad \forall n \in \mathcal{N}. \tag{12}$$

Proof. First, we consider the case when node n is not included in any shortest routes of users departing from their origins at potentials having reference time t. Because the potential on the node is not defined in Definition 5, we define the potential on the node as Eq. (12).

Next, we consider the case when node n is included in at least one of the shortest routes. We here divide origins into two groups: (i) Group A includes origins from which users departing at potentials have the shortest routes including node n (denoted by $\mathcal{N}^A_o$), and (ii) Group B includes the other origins (denoted by $\mathcal{N}^B_o$). For the origins in $\mathcal{N}^A_o$, $p_n(t) = u^*_n(o, p_o(t)),\ \forall o \in \mathcal{N}^A_o$ holds by the definition of unidirectional networks. Thus, it is sufficient for us to prove that the following equation holds for the origins in $\mathcal{N}^B_o$:

$$p_n(t) \leq u^*_n(o, p_o(t)), \quad \forall o \in \mathcal{N}^B_o. \tag{13}$$

We will prove it by contradiction. Suppose that the earliest arrival time at node n of a user departing from origin o in $\mathcal{N}_B$ is earlier than the potential, i.e., $u^*_n(o, p_o(t)) < p_n(t)$. Then, the user has route r_B that passes through the same nodes included in the way to destination d from node n along the route r^*_A of a user departing from origin o in $\mathcal{N}^A_o$, and the arrival times at these nodes are not later than the potentials, e.g., $u_n(o, p_o(t), r_B) \leq p^*_n(t)$ for node n. Because r_B is not the shortest route, there exists at least one node n' on which the DP principle does not hold at the downstream of node n along the route; the user can arrive at this node n' earlier by taking the shortest route r^*_B that satisfies the DP principle, i.e., $u_{n'}(o, p_o(t), r^*_B) < u_{n'}(o, p_o(t), r_B)$. Therefore, the following equation is obtained:

$$p^*_{n'}(t) = u_{n'}(o, p_o(t), r^*_B) < u_{n'}(o, p_o(t), r_B) \leq p^*_{n'}(t). \tag{14}$$

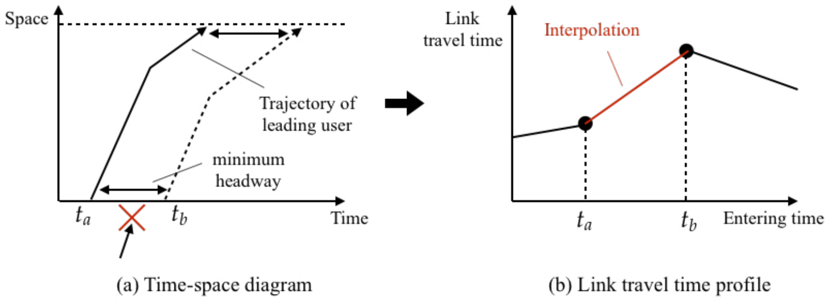


Fig. 1: The time-space diagram showing the relationship between trajectories of the leading user and following user and the link travel time profile on the link

This contradicts the definition of unidirectional networks. Hence, Eq. (13) is satisfied and Eq. (12) is obtained. □

Finally, by combining the results above, we have

$$t < t' \Rightarrow \; p_o(t) \leq p_o(t'), \qquad \forall o \in \mathcal{N}_o \tag{15}$$
$$\Rightarrow \; \min_o \{u_n^*(o, p_o(t))\} \leq \min_o \{u_n^*(o, p_o(t'))\} \qquad \forall n \in \mathcal{N} \tag{16}$$
$$\Rightarrow \; p_n(t) \leq p_n(t'). \tag{17}$$

Eq. (15) is obtained from Lemma 1. Eq. (16) is a direct consequence of Eq. (15), and Eq. (17) is obtained by combining Eq. (16) and Lemma 2. Therefore, Eq. (9) is obtained. □

While we implicitly assume that there exist *continuous* node potential functions, these are not the cases for the DUE problems with atomic users. On each node, there may exist some time intervals during which the node potential function cannot be defined, i.e., the function may become discontinuous. This is because there exist time intervals during which no user arrives at a node due to the following two reasons. First, a user cannot enter the link in the case that the user has less headway to its leading user than the minimum headway of the link (see Fig. 1(a)). In other words, there exist time intervals during which the link travel time function cannot be defined. Second, for the time interval during which no user departs from origins, Eq. (7) is not basically applicable because it is defined using the shortest routes of *users*. However, we can resolve these problems by introducing some appropriate interpolations. Specifically, we can resolve the first problem by introducing linear interpolations satisfying Eq. (6) (i.e., the FIFO principle) to the functions (see Fig. 1(b)). With regard to the second problem, what we should do is to introduce virtual users (vehicles) who are not players in the DUE game but are assigned to the network virtually to calculate node potentials. In both cases, we can define the continuous potential function without affecting the travel times of actual users.

3.2. *Existence of earliest non-assigned user in unidirectional network*

We are now ready to prove the existence of the earliest non-assigned user through the decomposition technique as follows:

Theorem 2. Consider a route choice profile $\mathbf{r} \in \mathcal{R}$ such that $\overline{\mathcal{P}}(\mathbf{r}) \neq \emptyset$ in a unidirectional nework. There exists at least one earliest non-assigned user.

Proof. We first consider the reference time t_j of user j such that $p_{o_j}(t_j) = s_j, \forall j \in \overline{\mathcal{P}}(\mathbf{r})$. Suppose that user i has the minimum reference time t_i. Then, there exists the shortest route $\overline{r}_i^*$ of user i such that

$$\begin{aligned} u_n^*(o_i, s_i; \mathbf{c}(\mathbf{r}_{-i})) &= p_n(t_i; \mathbf{c}(\mathbf{r}_{-i})), \\ &\leq p_n(t_j; \mathbf{c}(\mathbf{r}_{-j})), \\ &\leq u_n^*(o_j, s_j; \mathbf{c}(\mathbf{r}_{-j})), \qquad \forall n \in \mathcal{N}(\overline{r}_i^*) \cap \mathcal{N}_j, \forall j \in \overline{\mathcal{P}}(\mathbf{r}) \setminus \{i\}. \end{aligned}$$

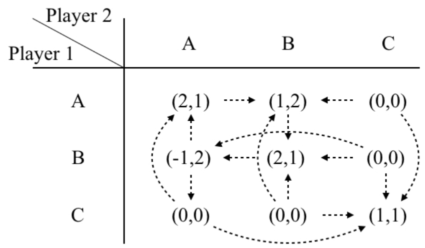


Fig. 2: Example of a two-player weakly acyclic game (we refer to Marden et al. (2009))

The first line of these equations means that user i takes the shortest route satisfying Eq. (7); the second line is obtained from Theorem 1, and the third line is a direct consequence of Eq.(12). This means that user i arrives at the nodes on route $\overline{r}_i^*$ at the same time with or earlier than any other non-assigned user, i.e., user i has the shortest route satisfying Eq. (5). Thus, user i is the earliest non-assigned user. □

4. Weakly acyclic games and convergence of DUE game in unidirectional network

In this section, we establish the relationship between the DUE game and the *weakly acyclic game* (Young, 1993). Specifically, we first introduce the definition and the concept of the weakly acyclic game. We then prove that the DUE game in a unidirectional network is a weakly acyclic game. After the proof, we investigate the convergence of the DUE game as the basis for investigating the stability in the next section.

Note that, hereinafter, we consider the situation that the DUE game is repeated on a day-to-day basis. Let $\tau = 1, 2, \ldots$ denote the successive time steps (i.e., day). The route choice and the utility of user $i \in \mathcal{P}$ at day τ are denoted by r_i^τ and $g_i(r_i^\tau; \mathbf{r}_{-i}^\tau)$, respectively. At each day, one user is randomly selected (we call this user *selected user*) and the user can change the current route to another route according to a behavioral rule, which is common to all users (i.e., day-to-day dynamics; we specify the dynamics later).

4.1. Definition of weakly acyclic games

The weakly acyclic game is characterized by the following *better response path*. Consider any game with a set of finite action profiles $\mathcal{R}$. A better response path is a sequence of action profiles $\mathbf{r}^1, \mathbf{r}^2, \ldots$ such that for each successive pair $\mathbf{r}^\tau, \mathbf{r}^{\tau+1}$, there is exactly one user i, who satisfies the following condition:

$$\begin{cases} r_i^\tau \neq r_i^{\tau+1}, & \text{s.t. } g_i(r_i^{\tau+1}; \mathbf{r}_{-i}^{\tau+1}) < g_i(r_i^\tau; \mathbf{r}_{-i}^\tau), \\ r_j^\tau = r_j^{\tau+1}, & \forall j \in \mathcal{P} \setminus \{i\}. \end{cases} \tag{18}$$

In other words, at each day, only one user changes the action to improve the arrival time at the destination.

The weakly acyclic game is then defined as follows (see also, Young (1993) and Marden et al. (2009)):

Definition 6. (**Weakly acyclic game**) A game is weakly acyclic if, and only if from every action profiles $\mathbf{r} \in \mathcal{R}$, there exists a better response path starting at $\mathbf{r}$ and ending at a (pure) Nash equilibrium of the game.

Note that whether a game is weakly acyclic or not is independent of the day-to-day dynamics applied to the game, i.e., only dependent on the structure of the game that determines the existence of such a better response path in the game. The weakly acyclic game is a generalized concept of the *potential games* (Monderer and Shapley, 1996). Specifically, a potential game is a class of a weakly acyclic game that does not have cycles in better response paths. However, both the weakly acyclic game and the potential game have a better response path ending at a Nash equilibrium.

An example of a two-player (user) weakly acyclic game is illustrated in Fig. 2. This figure shows the payoff matrix of the game and the dotted lines from a profile represent the better responses from the profile. From this figure, it can be seen that there exists a better response path ending at the Nash equilibrium of the game (C, C) from every profile.

For example, from profile (B,B), there exists a better response path $(B,B) \to (B,A) \to (C,A) \to (C,C)$. Note that a better response path that includes a cycle may exist (e.g., $(B,B) \to (B,A) \to (A,A) \to (A,B) \to (B,B)$ in the figure). However, a Nash equilibrium is reachable from the profiles in the cycle through the appropriate better response path in the weakly acyclic game.

4.2. A DUE game in a unidirectional network is a weakly acyclic game

Now we establish the relationship between the DUE game in a unidirectional network and the weakly acyclic game:

Theorem 3. A DUE game in a unidirectional network is a weakly acyclic game.

Proof. We prove the existence of better response paths ending at a Nash equilibrium from every action profile in a constructive manner. Specifically, we construct the algorithm that is guaranteed to converge to a Nash equilibrium from an arbitrary initial profile $\mathbf{r} \in \mathcal{R}$ by changing the users' route choices so that they improve the arrival times at the destinations.

We first introduce some definitions. The users of the game are divided into two groups: (i) Group A includes users who take ex-post shortest routes, and (ii) Group B includes users who do not. The sets of users in Group A and Group B are denoted by $\mathcal{P}_A$ and $\mathcal{P}_B$, respectively. We also denote by $\mathbf{r}_A$ and $\mathbf{r}_B$ the route choice profiles of Group A and Group B, respectively (i.e., $\mathbf{r} = (\mathbf{r}_A, \mathbf{r}_B)$). We then propose the following algorithm that finds a Nash equilibrium by transferring the user from $\mathcal{P}_B$ to $\mathcal{P}_A$ one by one:

0. **Initial setting**: Let $m = 0$, $(\mathcal{P}_A^m, \mathcal{P}_B^m) = (\emptyset, \mathcal{P})$ and $\mathbf{r}^m = (\mathbf{r}_A^m, \mathbf{r}_B^m) = (\emptyset, \mathbf{r})$. Here $\mathbf{r}$ is an initial route choice profile.
1. **Find an earliest non-assigned user**: Consider a new DUE game with only users in $\mathcal{P}_B^m$ but a route choice profile $\mathbf{r}_A^m$ is given fixed as a constraint. Then, regard the users in $\mathcal{P}_B^m$ as non-assigned users and search an earliest non-assigned user $i \in \mathcal{P}_B^m$ according to the criterion described in Section 2. We denote by g_i^* and $\overline{r}_i^*$ the earliest arrival time at the destination and the shortest route of user i, respectively.
2. **Update the route choice profile through better responses**: Compare the arrival times at the destination of a current route choice $g_i(r_i^m; \mathbf{r}_{-i}^m)$ with g_i^*. Then, there are exhaustive two cases:

   (a) If $g_i^* < g_i(r_i^m; \mathbf{r}_{-i}^m)$, then update the route choice of user i to $\overline{r}_i^*$, i.e., $r_i^{m+1} := \overline{r}_i^*$. The other users keep their current route choices: $r_j^{m+1} := r_j^m$ for all $j \in \mathcal{P}_B^m \setminus \{i\}$.

   (b) If $g_i^* = g_i(r_i^m; \mathbf{r}_{-i}^m)$, first search for the route choice profile $\mathbf{r}'_B$ of the users in $\mathcal{P}_B^m$ that satisfies the following conditions: (i) $\mathbf{r}'_B$ satisfies $g_i(r_i^m; \mathbf{r}_{-i}^m) < g_i(r'_i; \mathbf{r}'_{-i})$ where $\mathbf{r}' = (\mathbf{r}_A^m, \mathbf{r}'_B)$, and (ii) $\mathbf{r}'_B$ is reachable from $\mathbf{r}_B^m$ by a better response path.
   If there exists such a route choice profile, then update the route choice profile in the following way: $r_i^{m+1} := \overline{r}_i^*$ and $r_j^{m+1} := r'_j$ for all $j \in \mathcal{P}_B^m \setminus \{i\}$ (Case (b)-1). If such a route choice profile does not exist, then $r_j^{m+1} := r_j^m$ for all $j \in \mathcal{P}_B^m$ (Case (b)-2).

3. **Update the sets of users and judge the convergence**: Let $\mathcal{P}_A^{m+1} := \mathcal{P}_A^m \cup \{i\}$ and $\mathcal{P}_B^{m+1} := \mathcal{P}_B^m \setminus \{i\}$. If $\mathcal{P}_B^{m+1} \neq \emptyset$, let $m := m+1$ and go back Step 1. If $\mathcal{P}_B^{m+1} = \emptyset$, then terminate the algorithm; $\mathbf{r}^{m+1} = \mathbf{r}_A^{m+1}$ is a Nash equilibrium.

This algorithm finds a Nash equilibrium through a better response of the earliest non-assigned user to an ex-post shortest route. The role of each step in this algorithm is explained below.

In Step 1, an earliest non-assigned user is searched from $\mathcal{P}_B$. Note that an earliest non-assigned user always exists in a unidirectional network (see Theorem 2). Because the shortest route $\overline{r}_i^*$ satisfies Eq. (5), the users in $\mathcal{P}_B^m$ cannot overtake user i when the user takes $\overline{r}_i^*$. Therefore, the route choice of user i becomes the ex-post shortest route when user i can change the current route r_i^m to $\overline{r}_i^*$ by a better response.

Step 2 checks whether the current route r_i^m of the selected user i is an ex-post shortest route or not; if not, then the algorithm changes the route choice of i to the ex-post shortest route $\overline{r}_i^*$ through better responses. In case (a), it is clear that user i does not take an ex-post shortest route since the arrival time by taking r_i^m is later than that by taking $\overline{r}_i^*$; thus the route choice of user i is changed to $\overline{r}_i^*$ through a better response of the user.

In case (b), it is not obvious that r_i^m is an ex-post shortest route although the destination arrival time by taking the route is the same with that by taking $\overline{r}_i^*$. This is because the arrival time of user i by taking r_i^m may increase due to overtaking by other users in $\mathcal{P}_B^m$ through their better responses to a route choice profile $\mathbf{r}'_B$. Therefore, the algorithm checks the existence of such a better response path[8]. If it exists (Case (b)-1), then r_i^m is not an ex-post shortest route. In this case, first, the route choices of users in $\mathcal{P}_B^m$ are changed to $\mathbf{r}'_B$; then, because the arrival time of user i increases due to the better responses, the route choice of user i is changed to the ex-post shortest route $\overline{r}_i^*$ through a better response of the user. If not (Case (b)-2), it is guaranteed that r_i^m is the ex-post shortest route because its arrival time is already minimized (i.e., $g_i^* = g_i(r_i^m; \mathbf{r}_{-i}^m)$); there is no need to change the route choices of users in $\mathcal{P}_B^m$.

Finally, in Step 3, the algorithm transfers user i from $\mathcal{P}_B$ to $\mathcal{P}_A$. In this algorithm, the route choices in the users in $\mathcal{P}_A^m$ remain ex-post shortest routes because the arrival times at the destinations of the users in $\mathcal{P}_A^m$ are independent of the route choices of the users in $\mathcal{P}_B^m$.

As a result, this algorithm can find a Nash equilibrium at exactly $|\mathcal{P}|$ iterations through the better response of each user when the existence of the earliest non-assigned user is guaranteed. The existence in the DUE game in a unidirectional network is guaranteed as mentioned above. This means that there exists a better response path ending at a Nash equilibrium from an arbitrary initial profile in the DUE game. □

Note that this algorithm is applicable to a DUE game in which the existence of the earliest non-assigned user is guaranteed even when the network is not unidirectional. Thus, the following corollary is obtained:

Corollary 1. The DUE game in which the earliest non-assigned user exists is the weakly acyclic game.

4.3. Convergence of DUE game in unidirectional network

We present an investigation of the convergence of a day-to-day dynamics in a DUE game. Although the convergence property of the dynamics is not a sufficient condition for equilibria to be stable, this property is useful for investigating the stochastic stability as we will show in Section 5. Here, we consider the following two representative dynamics: *better response dynamics* (Hart and Mas-Colell, 2000) and *best response dynamics* (Blume, 1993).

4.3.1. Better response dynamics

We first show the convergence of the better response dynamics. At each day τ (> 0), one user $i \in \mathcal{P}$ is randomly chosen and allowed to change the current route choice. The other users must repeat their current route choices at the day. Then, user i changes route r_i^τ to $r_i^{\tau+1}$ such that the user arrives at the destination strictly earlier than the previous day, i.e., $g_i(r_i^{\tau+1}, \mathbf{r}_{-i}^\tau) < g_i(r_i^\tau, \mathbf{r}_{-i}^\tau)$. If the user does not have the route satisfying the condition, the user does not change the route choice. The route choice set of better response of user i is denoted by $D_i(\mathbf{r})$, and

$$D_i(\mathbf{r}^\tau) := \left\{ r_i^* \in \mathcal{R}_i \text{ s.t. } g_i(r_i^*, \mathbf{r}_{-i}^\tau) < g_i(r_i^\tau, \mathbf{r}_{-i}^\tau) \right\}. \tag{19}$$

It is obvious that a Nash equilibrium becomes a rest point of the better response dynamics. Thus, Eq. (19) is regarded as one of the natural behavioral rules of the atomic user that satisfies the *positive correlation* and *Nash stationary*, which are the criteria proposed by Sandholm (2010).

We can obtain the convergence of this dynamics in the DUE game in a unidirectional network as a direct consequence of Theorem 3 as follows: Because the DUE game is the weakly acyclic game, there exists at least one better response path connecting to a Nash equilibrium from every profile. As $\tau \to \infty$, the probability that such a better response path is selected becomes 1, i.e., the better response dynamics in the DUE game almost surely converges to a Nash equilibrium. Thus, the following proposition is established:

Proposition 1. In any DUE game in a unidirectional network, the route choice profile $\mathbf{r}$ generated by the better response dynamics from an arbitrary initial profile converges to a Nash equilibrium almost surely as $\tau \to \infty$.

Proof. Young (2004) proved the convergence of the better response dynamics in weakly acyclic games. □

[8] Note that the algorithm shown here is a pseudo algorithm, and an additional procedure have to be specified when this algorithm is implemented actually. However, it is possible to conduct this step in principle since the state space $\mathcal{R}$ is finite. Thus, this does not matter in this proof.

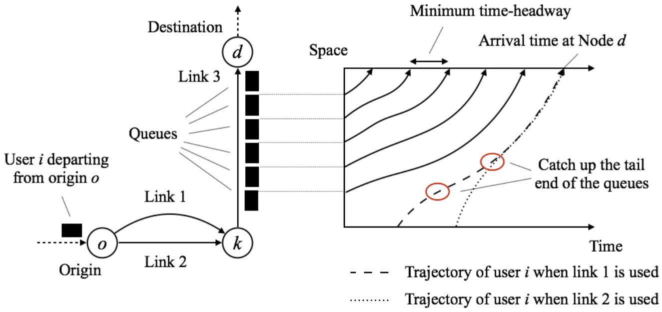


Fig. 3: Illustrated network and time-space diagram showing the trajectories of the user on link having queues

4.3.2. *Best response dynamics*

We then introduce the best response dynamics (Blume, 1993). In this dynamics, the selected user i chooses the route that has the minimum arrival time at the destination. In other words, the user i chooses the route randomly from the following route choice set $B_i(\mathbf{r}^\tau)$:

$$B_i(\mathbf{r}^\tau) := \left\{ r_i^* \in \mathcal{R}_i \text{ s.t. } g_i(r_i^*, \mathbf{r}_{-i}^\tau) \le \min_{r \in \mathcal{R}_i} g_i(r, \mathbf{r}_{-i}^\tau) \right\}. \tag{20}$$

In the best response dynamics, a *strict* Nash equilibrium becomes a rest point of the dynamics (Alós-Ferrer and Netzer, 2017). The reason why a *non-strict* Nash equilibrium does not become the rest point is that the user taking an optimal choice in an equilibrium is allowed to change the choice to another optimal choice whose utility is the same; on this point, the best response dynamics is widely different from the better response dynamics in which the user is not allowed to change its choice to the other optimal choice. Such a change of the user's choice could affect the optimal choices of other users and the state could deviate from the equilibrium due to the best responses of them. Thus, to prove the convergence of the best response dynamics, the DUE game should have a strict Nash equilibrium.

However, in the DUE game, the strictness of a Nash equilibrium cannot be guaranteed in general. This is because the uniqueness of the shortest route of a user may be lost when there exist queues in the network. For example, consider the network shown in Fig. 3. This network has one origin and one destination and there exist two routes passing link 1 or 2. Suppose that in an equilibrium state, link 3 has queues and a user can catch up the tail end of the queue before the queue dissipates by using either route; in addition, users departing at a time later than this user do not overtake this user. Then, we can see that the destination arrival times of both routes become the same because the earliest possible departure time from link 3 is restricted to the departure time of the leading user as shown in the figure. This means that multiple ex-post shortest routes exist (i.e., multiple best response actions in Eq. (1) for the user), and the strictness of equilibrium is lost. Consequently, we can conclude that the convergence of the best response dynamics in the DUE game is not guaranteed in general.

From the discussion in the previous and this sections, it is evident that the *strict improvement of users' utilities in a dynamics* or the *strictness of equilibrium* plays a central role in guaranteeing the convergence of the dynamics. Furthermore, because the strict Nash equilibrium may not exist in the DUE game due to queues, the better response dynamics that has the strict improvement property is more desirable than the best response dynamics in that the convergence is guaranteed in general; this is an interesting effect of queues in that the strictness of equilibrium could be lost. Moreover, the difference in convergence properties between these dynamics will affect the *stochastic stability* of the dynamics, as will be shown in the next section.

5. Stochastic stability of DUE game

5.1. *Definition of stochastic stability*

Young (1993) introduced the concept of stochastic stability to investigate the asymptotic behavior as follows: *Over the long run, states that are not stochastically stable will be observed infrequently compared to states that are stable,*

provided that the probability of mistakes ϵ is small. To introduce its formal definition, we first consider a finite state Markov chain over the state space $\mathcal{R}$ generated by the dynamics without perturbation (e.g., better response dynamics), referred as the *unperturbed Markov chain*. Let $\mathbf{P}^0$ be the probability transition matrix of the Markov chain. $\mathbf{P}^0_{\mathbf{rr}'}$ is the transition probability from state $\mathbf{r}$ to state $\mathbf{r}'$. Then, consider the *regular perturbed Markov chain* generated by the dynamics, which is continually perturbed by small mutations or mistakes. In this Markov chain, the selected user randomly chooses the action from the set of all available actions with positive probability characterized by parameter ϵ. Let $\mathbf{P}^\epsilon$ be the probability transition matrix of the perturbed Markov chain. As the perturbation becomes small, such a random choice probability becomes zero and the probability transition matrix of the perturbed Markov chain becomes the same with the unperturbed one, i.e., $\lim_{\epsilon\to 0}\mathbf{P}^\epsilon = \mathbf{P}^0$.

For any $\epsilon > 0$, the perturbed Markov chain is aperiodic and irreducible, i.e., the Markov chain is ergodic. This means that the perturbed Markov chain has a unique stationary distribution $\boldsymbol{\mu}^\epsilon$ satisfying $\boldsymbol{\mu}^\epsilon\mathbf{P}^\epsilon = \boldsymbol{\mu}^\epsilon$. This stationary distribution gives the observation probability of each state when the process with the perturbation runs for a very long time. In addition, it has been shown that such a stationary distribution converges to one of the stationary distribution of the unperturbed Markov chain as $\epsilon \to 0$ (Young, 1993). Then, the stochastic stability (Young, 1993) is defined as follows:

Definition 7. (**Stochastic stability**) A state $\mathbf{r} \in \mathcal{R}$ is *stochastically stable* relative to a perturbed Markov chain if $\lim_{\epsilon\to 0}\boldsymbol{\mu}^\epsilon_{\mathbf{r}} > 0$.

5.2. Stochastic stability of DUE game in unidirectional network

We will now investigate the stochastic stability of a DUE game in a unidirectional network. Here, we consider the following two perturbed dynamics: *perturbed better response dynamics* and *perturbed best response dynamics*. Particularly in the latter, we discuss the stability of the logit response dynamics (Blume, 1993; Marden and Shamma, 2012; Alós-Ferrer and Netzer, 2017), which is one of the perturbed best response dynamics and is widely used in traffic assignments (e.g., Miyagi et al., 2013).

5.2.1. Perturbed better response dynamics

In the perturbed better response dynamics, one user $i \in \mathcal{P}$ is randomly chosen and allowed to change the current route choice at each day, as with the case without perturbation. However, with the probability $p_i(\epsilon)$ characterized by ϵ, the selected user chooses a route randomly from $\mathcal{R}_i$ instead of the better response. Then, the transition probability from state $\mathbf{r} = (r_i, \mathbf{r}_{-i})$ to $\mathbf{r}' = (r'_i, \mathbf{r}_{-i})$ of this dynamics is described as follows:

$$\mathbf{P}^\epsilon_{\mathbf{rr}'} = (1 - p_i(\epsilon))\mathbf{P}^0_{\mathbf{rr}'} + \frac{1}{n}\cdot p_i(\epsilon)\cdot q_i(r'_i;\mathbf{r}_{-i}) \tag{21}$$

where $\mathbf{P}^0_{\mathbf{rr}'}$ represents the transition probability in better response dynamics without perturbation. $q_i(r'_i;\mathbf{r}_{-i})$ is the transition probability to route r'_i such that $\sum_{r'_i\in\mathcal{R}_i} q_i(r'_i;\mathbf{r}_{-i}) = 1$. $p_i(\epsilon)$ and $q_i(r'_i;\mathbf{r}_{-i})$ are dependent on the applied perturbation. For example, when we assume the perturbation such that the user randomly chooses the route regardless of the destination arrival time, then these probabilities are determined as follows: $p_i(\epsilon) = \epsilon$, $q_i(r'_i;\mathbf{r}_{-i}) = 1/|\mathcal{R}_i|$.

Regarding this dynamics, the stochastic stability of DUE is presented as follows:

Proposition 2. Consider a DUE game in a unidirectional network. Then, there exists a stochastically stable equilibrium of the perturbed better response dynamics.

Proof. Young (1993) showed the following theorem that provides a criterion for determining the stochastically stable states:

Theorem 4. (**Young (1993), Theorem 4**) Consider an unperturbed Markov chain on the finite state space $\mathcal{R}$ with recurrent communication classes $r_1,\ldots,r_J$, and a regular perturbation of the Markov chain that has a unique stationary distribution $\boldsymbol{\mu}^\epsilon$ for every small positive ϵ. Then, as $\epsilon \to 0$, $\boldsymbol{\mu}^\epsilon$ converges to a stationary distribution $\boldsymbol{\mu}^0$ of the

unperturbed Markov chain. In addition, the stochastically stable states are contained in the recurrent classes with the minimum stochastic potential[9].

Then, the theorem can be proved by utilizing this theorem and the property of the better response dynamics in a weakly acyclic game as follows. Because a DUE game in a unidirectional network is a weakly acyclic game, the recurrent classes of the unperturbed Markov chain correspond one on one to the rest points of the better response dynamics. In addition, the rest points correspond to Nash equilibria of the game. Thus, from Young's theorem, it follows that the stochastically stable states are contained in the set of Nash equilibria. Therefore, there exists a stochastically stable equilibrium of the perturbed better response dynamics. □

Note that the proposed approach can be applied to a day-to-day dynamics under multiple users change their route choices simultaneously. This is because for any game that is weakly acyclic game, the rest points of such a synchronous better response dynamics are precisely a set of Nash equilibria; then, we can prove the existence of a stochastically stable equilibrium of the perturbed synchronous better response dynamics by utilizing Young's theorem (Marden and Shamma, 2012).

5.2.2. Perturbed best response dynamics

Next, we investigate the stochastic stability of the perturbed best response dynamics. One of the most known perturbed best response dynamics is the logit response dynamics. In this dynamics, selected user i chooses the route choice r_i according to the probability $p_i(r_i, \mathbf{r}_{-i})$ given by

$$p_i(r_i; \mathbf{r}_{-i}) = \frac{\exp(-\beta g_i(r_i, \mathbf{r}_{-i}))}{\sum_{r_i \in \mathcal{R}_i} \exp(-\beta g_i(r_i, \mathbf{r}_{-i}))}. \tag{22}$$

where $0 < \beta < \infty$ measures the degree of noise in the best response. The logit response dynamics converges to the best response dynamics when $\beta \to \infty$.

The stochastic stability of this dynamics is proved in Theorem 1 by Alós-Ferrer and Netzer (2017) as follows: *Consider a weakly acyclic game in which the best response dynamics converges to a strict Nash equilibrium. Then the set of stochastically stable states of the logit response dynamics is contained in the set of strict Nash equilibria*. This is the application of Young's theorem, which utilizes the fact that the rest point of the best response dynamics is the strict Nash equilibrium. However, as mentioned above, the existence of the strict Nash equilibrium is not guaranteed in general due to the existence of queues. Thus, we can conclude that the existence of the stochastically stable equilibrium of the logit response dynamics is not guaranteed in general, either.

6. Numerical experiments

We finally conducted numerical experiments to demonstrate the convergence and stochastic stability of the DUE game in a unidirectional network. Specifically, we first show the convergence of the better response dynamics. We here also compare the convergence speed of the dynamics in the DUE game with the physical-queue model and point-queue model. We then demonstrate the difference in theoretical properties of stochastically stable equilibria of the perturbed better response dynamics and logit response dynamics.

6.1. Settings

We considered a unidirectional network with many-to-many OD, which is a modification of the Nguyen-Dupuis network (as shown in Fig. 4). Nodes $\{o_1, o_2\}$ are origins, and nodes $\{d_1, d_2\}$ are destinations. The physical condition of each link (e.g., free-flow travel time and capacity) are summarized in Table. 1. Each link has a bottleneck section with a bottleneck capacity at the end of the link. The network has 4 routes for each destination, respectively. These routes are numbered as shown in the figure. The number of users departing from each origin for each destination is 50, and the total number of users is 200. Each user from each origin departs with a fixed time headway (we will specify later).

[9] We omit the definition of the stochastic potential here because this is not related to the discussion in this paper (see Young (1993) for details).

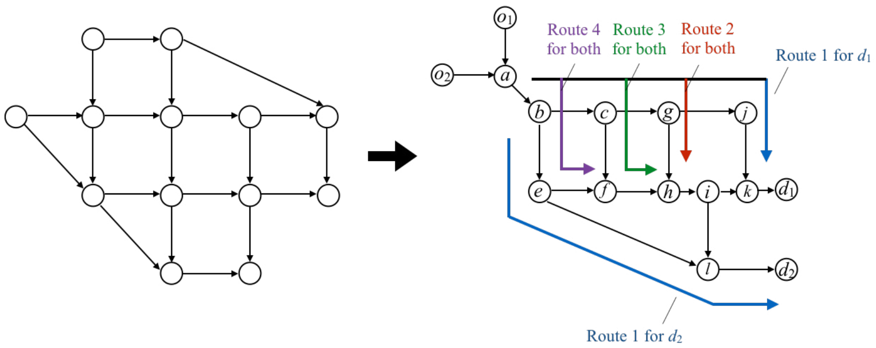


Fig. 4: Left: Original Nguyen Dupuis network. Right: Modification to the unidirectional network

Table 1: Physical condition of links (FFTT: Free Flow Travel Time, BC: Bottleneck Capacity, SF: Saturation Flow)

Link	FFTT [sec]	BN [veh/sec]	SF [veh/sec]	Link	FFTT [sec]	BN [veh/sec]	SF [veh/sec]
(o_1, a)	14	2	2	(f, h)	12	1.5	1.5
(o_2, a)	18	2	2	(g, h)	18	0.17	0.5
(a, b)	6	2	2	(g, j)	6	0.2	0.5
(b, c)	6	2	2	(h, i)	9	0.67	83
(b, e)	18	1.5	1.5	(i, k)	12	0.27	0.67
(c, f)	26	1.5	1.5	(i, l)	6	0.2	0.5
(c, g)	10	1.5	1.5	(j, k)	6	0.67	1
(e, f)	20	1.5	1.5	(k, d_1)	6	0.67	1
(e, l)	42	0.27	0.5	(l, d_2)	6	0.67	1

Traffic dynamics within the network is simulated using a mesoscopic LWR model proposed by Leclercq and Bécarie (2012). This simulator provides the event time when each user crosses the specific points of the network (e.g., nodes); then according to the event time, the trajectory of each user is calculated based on the dynamic loading model. We employ the simplified car following model by Newell (2002) as the dynamic loading model.

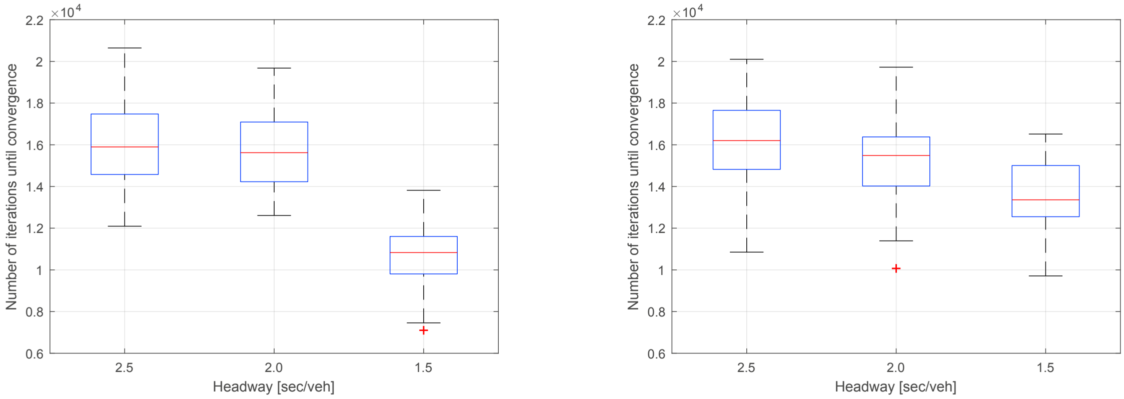
6.2. Results

6.2.1. Convergence of better response dynamics

In this experiments, we will see whether the difference in the dynamic loading model affects the convergence speed or not. Here, we consider the different queue models, the physical queue and point queue, by changing a parameter in the car-following model. We also consider three demand levels by changing the headway: with 2.5[sec] (referred to as low), 2.0[sec] (medium), 1.5[sec] (high). For each case, we iterate the better response until convergence to a Nash equilibrium and repeat this process 50 times; we then compare the number of iterations until convergence between the cases. Note that the initial route choice of each user is set as the shortest-distance route.

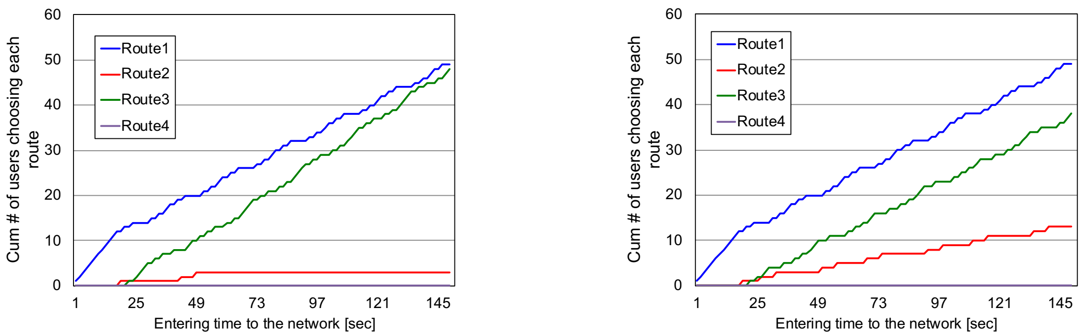
Fig. 5 displays the comparison result of the convergence speed. The horizontal axis represents the demand level, and the vertical axis represents the number of iterations of the better response until convergence. From this figure, we observe that the number of better response required to reach a Nash equilibrium increases as the demand level reduces, and the point queue model is employed, i.e., a large number of iterations are required to fix the users route choices to their ex-post shortest routes. This implies that queue spillbacks reduce the number of candidates of the better response in this particular scenario.

To investigate the mechanism, we examine the route choice pattern at equilibrium. Fig. 6 shows the cumulative number of users choosing each route for d_1 with the physical and point queue models. First, from Fig. 6a, we observe that route 2 is unused early in the simulation with the physical queue model. This is because the queues on link (j, g) spillback to link (c, g), and the travel time of route 2 increases. This implies that the candidates of the better response for most of the users until convergence are route 1 and 3. On the other hand, from Fig. 6b, we observe that users continue using route 2 in all times of the simulation. This is because there is no queue spillback and route 2 remains to be the shortest route. As a result, there are three candidates for the better response. Hence, in this particular scenario,



(a) Physical queue (b) Point queue

Fig. 5: Number of iterations of the better response until convergence to a Nash equilibrium



(a) Physical queue (b) Point queue

Fig. 6: Cumulative number of users whose destination is d_1 choosing each route

we observe that the probability of selecting the ex-post shortest route becomes large due to queue spillbacks, i.e., the queue spillover can reduce the number of better responses required to reach a Nash equilibrium. However, the effect of the queue and the convergence speed may largely depend on the initial conditions and numerical settings. A systematic numerical experiment should be conducted for different types of network settings and the effect should be investigated.

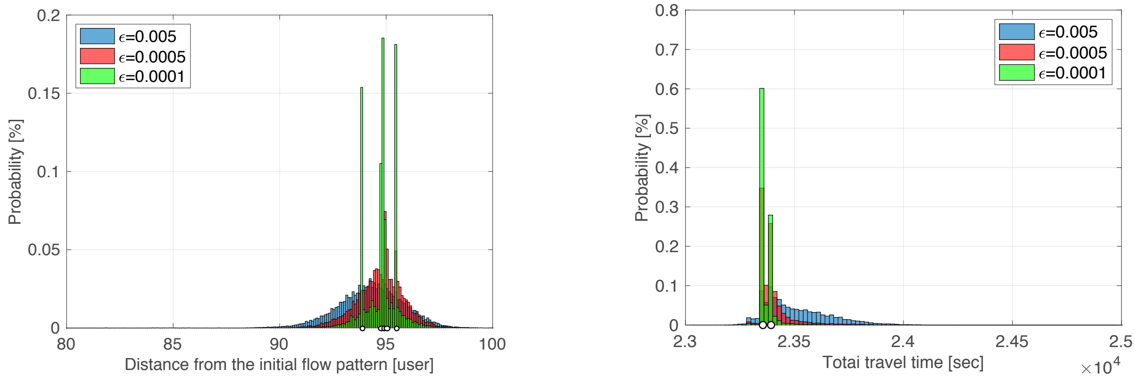
6.2.2. Stationary distribution of perturbed dynamics

Next, we demonstrate the difference in the theoretical properties of stochastically stable equilibria of perturbed better/best response dynamics by investigating the stationary distributions. In this experiment, we only consider the high demand level with the physical queue model. The perturbation of the better response dynamics is set such that $p_i(\epsilon) = \epsilon$ and $q_i(r'_i; \mathbf{r}_{-i}) = 1/|\mathcal{R}_i|$ for user i (i.e., the selected user randomly chooses the route regardless of the arrival time). Each response is iterated 200,000 times with the three perturbation levels: for the perturbed better response, $\epsilon = 0.005, 0.0005$, and 0.0001; for the logit response, $\beta = 0.1, 1$, and 100.

For each dynamics with each perturbation level, we obtain two frequency distributions of the users' route choice pattern and of the total travel time. Particularly, in the former, the state of each day $z(\tau)$ is defined as the Euclidean distance from the initial flow pattern (the shortest-distance route choice pattern) as follows:

$$z(\tau) = \left[\sum_{r \in \mathcal{R}} (f^r(\tau) - f^r(0))^2 \right]^{1/2}, \tag{23}$$

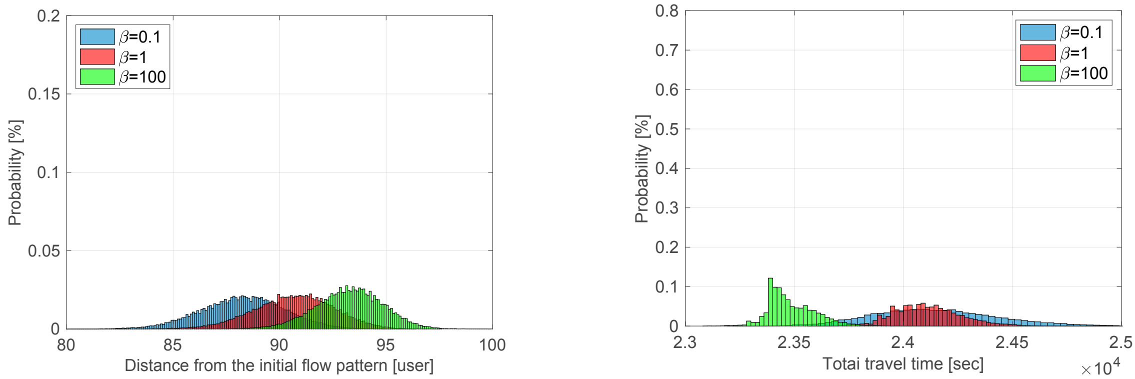
where $f^r(\tau)$ is the number of users who choose route r in day τ.



(a) Route choice patterns

(b) Total travel times

Fig. 7: Stationary distributions of the perturbed better response dynamics (white circles show Nash equilibriums)



(a) Route choice patterns

(b) Total travel times

Fig. 8: Stationary distributions of the logit response dynamics

We first show the stationary distributions of the perturbed better response dynamics in Fig.7. In each figure, the horizontal axis shows states, and the vertical axis shows the frequency of these states. From Fig. 7a, we observe that the distribution tends to converge to multiple Nash equilibria according to the decrease in the perturbation level. Meanwhile, from Fig. 7b, we also observe that the total travel times converge to almost the same value although there is a slight difference. This implies that the link travel time pattern is also almost the same and expected to converge to the same value as the number of users becomes large.

Next, we show the distributions of the logit response dynamics in Fig. 8a and Fig. 8b. As can be seen from these figures, the distributions do not converge to a particular state. In addition, a Nash equilibrium is not reached during the iteration of this dynamics. These results imply that the stochastically stable equilibrium may not exist in these settings.

7. Conclusion

In this study, we examined the stability of DUE in a unidirectional network. The presented approach consists of three key concepts: the decomposition technique of DUE assignments, the weakly acyclc game, and the asymptotic analysis of stationary distribution of perturbed dynamics. This synthesizing approach enables us to examine the theoretical properties of DUE without requiring the monotonicity of route travel time function. Specifically, after formulating a DUE assignment as a strategic game, we first proved the existence of the earliest non-assigned user in a unidirectional network, which implies that there exists an order of assigning users one by one to the network

for ensuring an equilibrium. With this ordering property, we proved that the DUE game in a unidirectional network is the weakly acyclic game, which guarantees the convergence of better response dynamics to an equilibrium. We then established the existence of a stochastically stable equilibrium of the better response dynamics. Furthermore, we found that the strict improvement of user's utility is the important condition for this existence property in the DUE game because the strictness of equilibrium could be lost due to queues. Finally, we conducted numerical experiments to demonstrate the convergence and stability of the DUE in a unidirectional network.

While we showed the stability of the DUE in the unidirecational network, how the flow and cost patterns arise at the stable equilibrium is largely unknown. In particular, if there are multiple equilibria, it is important to examine which equilibrium is selected by the dynamics (i.e., an equilibrium selection) and how to characterize this equilibrium. Also, if there are efficient and inefficient equilibria, it is interesting to design an incentive scheme to make the resulting day-to-day dynamics to converge to the former equilibrium (or system optimal assignments). These analyses may be challenging even for the DUE game in the unidirectional networks because the state space is large and there are complex interactions among physical queues. Nevertheless, these seem fruitful topics for establishing a distributed control of dynamic transportation networks. Furthermore, we are interested in exploring the applicability of the proposed approach to the other types of networks and to the DUE assginemtns with departure time choices.

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Appendix A. Proof of Lemma 1

Consider an origin $o \in \mathcal{N}_o$ and the shortest routes $r^*(t)$ and $r^*(t')$ departing from this origin at potentials $p_o(t)$ and $p_o(t')$. Consider also node $n \in \mathcal{N}(r^*(t)) \cap \mathcal{N}(r^*(t'))$. Hereafter, we first prove Eq. (11) and then obtain Eq. (10).

The first line of Eq. (11) is simply obtained from the FIFO principle of a dynamic loading model, as also shown by Iryo and Smith (2017). Specifically, the following relationships hold because of the FIFO principle (i.e., no overtaking on the same route):

$$\begin{cases} p_o(t) < p_o(t') \Rightarrow p_n(t) \leq p_n(t') \\ p_o(t) = p_o(t') \Rightarrow p_n(t) = p_n(t') \end{cases} . \tag{A.1}$$

This means that, for a given order of the potentials on the origin, the order of the potentials on node n is determined in a *forward manner* along the traffic flow on the shortest routes.

We next prove the second line of Eq. (11) in a *backward manner* along the traffic flow. First, if there is a strict order between two potentials on node n, it is apparent that the order between two potentials on node o also is strict owing to the FIFO principle. That is,

$$p_n(t) < p_n(t') \Rightarrow p_o(t) < p_o(t'). \tag{A.2}$$

Eq. (A.2) shows that a user arriving at a node earlier must depart from the origin earlier. This suggests that the the order of the potentials on the origin can be determined if there exists at least one node satisfying the following property: the node is included in the shortest routes from the origin, and the order of the potentials on the node is strict.

On the other hand, however, when the potentials on all nodes that are included the shortest routes from the origin are the same (i.e., $p_n(t) = p_n(t')$), there are cases where the order between $p_o(t)$ and $p_o(t')$ is not determined uniquely. The cases are caused by the existence of a *zero-flow congested link*. The zero-flow congested link is the link that has queues with the time interval in which there are no incoming flows. On the link, the change in the travel time function per unit time is equal to -1 during a certain time interval, i.e., $\frac{c_l(t')-c_l(t)}{t'-t} = -1$. This means that the leaving time of a user from the link becomes the same regardless of the entering time of a user who enters during the interval.

We demonstrate this issue using an example shown in Fig. A.9. The left side of the figure shows the network with three origin $\{o_1, o_2$, and $o_{REF}\}$ and one destination d. o_{REF} is chosen as the reference node. Links (o_{REF}, d) and (n, d) are

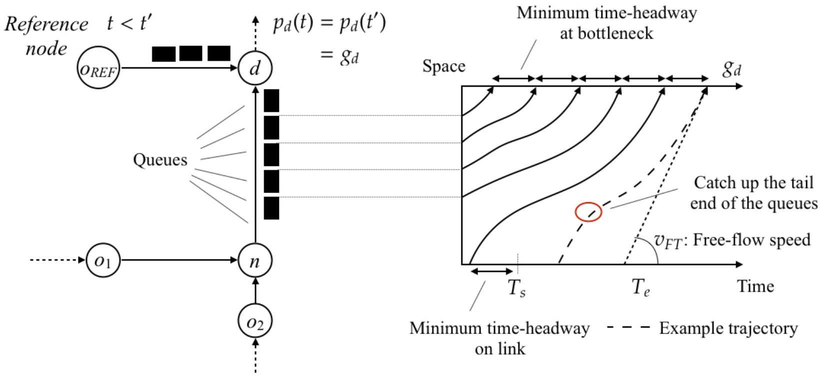

Fig. A.9: Example network that has a zero-flow congested link and the time-space diagram showing the trajectories of users on link (n, d)

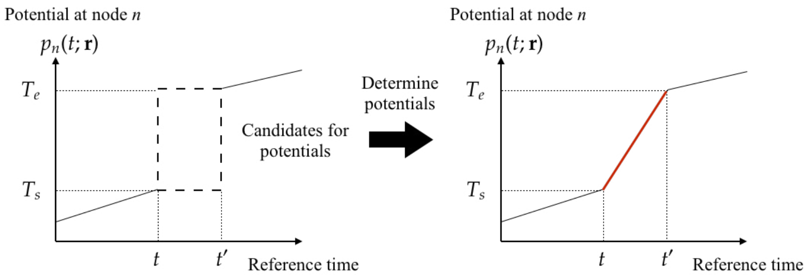

Fig. A.10: Arbitrary relation between potentials on node n and one example of determining the relation

the zero-flow congested links, which have queues represented as black rectangles. The right side of the figure shows the time-space diagram on link (n, d). This diagram shows that the leaving time from the link becomes g_d regardless of the entering time of a user who enters during the time interval $[T_s, T_e]$ due to the existence of queues. Also, the link (o_{REF}, d) is a zero-flow congested link and then the potentials on node d having different reference times t and t' can be the same with g_d, i.e., $p_d(t) = p_d(t') = g_d$. In this situation, we cannot uniquely determine the order between $p_n(t)$ and $p_n(t')$ in addition to the value of each potential, as shown in the left side of Fig. A.10.

However, the physical condition that should be satisfied here is that $p_n(t), p_n(t') \in [T_s, T_e]$; as long as this condition is satified, the correspondence between the reference times and the potential times at node n is arbitrary. In other words, in the left side of Fig. A.10, we can draw an arbitrary line on the area surrounded by the dotted lines. Thus, without loss of generality, we can set the correspondence such that satisfies the following condition:

$$t < t' \wedge p_n(t) = p_n(t') \Rightarrow p_o(t) \leq p_o(t') \qquad o \in \{o_1, o_2\} \tag{A.3}$$

Eq. (A.3) shows the condition for ensuring that the monotonic relation between the reference times and the potentials. An example of the correspondence that satisfies this condition is shown in the right side of Fig. A.10.

As a result, summarizing Eq. (A.2) and Eq. (A.3), for $t < t'$, we obtain the second line of Eq. (11):

$$p_n(t) \leq p_n(t') \Rightarrow p_o(t) \leq p_o(t'), \quad \forall o \in \mathcal{N}_o, \forall n \in \mathcal{N}(r^*(t)) \cap \mathcal{N}(r^*(t')). \tag{A.4}$$

Now, we are in a position to prove Eq. (10). First, for $t < t'$, we can determine the order of the potentials on node n that is included in the shortest routes from the reference node, by using Eq. (A.1) as follows: $t < t' \Rightarrow p_n(t) \leq p_n(t')$. Then, we can also determine the order of the potentials on origin o from which the shortest routes include this node n by using Eq. (A.4) as follows: $p_n(t) \leq p_n(t') \Rightarrow p_o(t) \leq p_o(t')$. Therefore, by recursively applying Eqs. (A.1) and (A.4), we can obtain

$$t < t' \Rightarrow p_o(t) \leq p_o(t'), \quad \forall o \in \mathcal{N}_o. \tag{A.5}$$

Thus, Eq. (10) is proved. □

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