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A New Cell Transmission Model with Priority Vehicles and Special Lanes

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Abstract

Daganzo (1997) proposed a kinematic wave model of a traffic system with priority vehicles and special lanes, where the priority vehicles can use both regular and special lanes, but the regular vehicles can only use the regular lanes. Further in Daganzo et al. (1997), the Incremental Transfer (IT) principle was applied to devise a Cell Transmission Model to numerically solve the model.

In this paper we first derive the fundamental diagram of a traffic system with priority vehicles and special lanes based on Wardrop's user equilibrium principle and then show after transforming the variables that there exist two auxiliary Lighthill-Whitham-Richards model, which can be used to define the demand and supply functions. We present a new junction flux function for the Cell Transmission Model, which is simpler than the IT principle, and more importantly, is a lane-based formulation. Comparing the proposed fluxes with the Godunov (or IT principle) ones analytically, we identify the cases that they are different. Nevertheless, with numerical experiments, we demonstrate that they are consistent in all the cases in the sense that the stationary states of both models are identical. Finally, we show numerical examples that demonstrate the IT principle is not invariant by using a non-triangular fundamental diagram.

Keywords: kinematic wave model; cell transmission model; multi-lane; multi-class

1. Introduction

Congested traffic on a road with special lanes such as High Occupancy Vehicle (HOV) lanes and segregated off-ramp lanes can behave much more complex than normal road sections because the vehicles who use those lanes interacts others. For describing such complex phenomena, Daganzo (1997) proposed a kinematic wave model of a traffic system with priority vehicles and special lanes, where the priority vehicles can use both regular and special lanes, but the regular vehicles can only use the regular lanes. From Wardrop's user equilibrium principle, the priority vehicles' speeds are never lower than the regular vehicles'. From this observation, the fundamental diagrams for both priority and regular vehicles are derived. An important insight is that traffic can be in 1-pipe or 2-pipe regimes: in the

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1-pipe regime, priority vehicles use both special and regular lanes, and have the same speeds as regular vehicles; but in the 2-pipe regime, priority vehicles only use the special lanes, and their speeds are not lower than the regular vehicles. Then from the conservation of both special and regular vehicles, a system of two coupled hyperbolic conservation laws is derived, and its Riemann problem was analytically solved according to a number of entropy conditions. However, the entropy conditions are quite heuristic, and the resulting Godunov flux formula may be true only for triangular fundamental diagrams.

Further in Daganzo et al. (1997), the Incremental Transfer (IT) principle was applied to devise a Cell Transmission Model (CTM) (Daganzo, 1995; Lebacque, 1996) to numerically solve the model. In this model, the fluxes across a boundary between an upstream and a downstream cells are calculated from the upstream demands of both priority and regular vehicles and the downstream supplies of both special and regular lanes. The Wardop's user equilibrium principle was also used to separate two cases: when the ratio of the priority vehicles' demand is higher than that of the special lanes' supply, the priority vehicles use both types of lanes, and their flux is proportional to the demand; otherwise, the priority vehicles only use the special lanes, and the two traffic flows are separated. The reference also analytically proved the equivalence of the IT and Godunov fluxes. However, as we will demonstrate later, this equivalence may not necessary hold for non-triangular fundamental diagrams.

All of the aforementioned limitations, both theoretical and numerical, could hinder the further extension of the model for more complicated situations, e.g., with High Occupancy Toll (HOT) lanes, where the regular vehicles (single occupancy vehicles) are also allowed to use the special lanes (HOT lanes) subject to tolls.

As a first step to overcome these limitations, in this paper we present a new formulation of Daganzo's kinematic wave model and the corresponding CTM. Specifically, after deriving the fundamental diagram of a traffic system with priority vehicles and special lanes and transforming the variables, we first show that there exist two auxiliary Lighthill-Whitham-Richards (LWR) model (Lighthill and Whitham, 1955; Richards, 1956), which can be used to define the demand and supply functions. We then present a new junction flux function for the CTM that is simpler than the IT principle. Comparing the proposed fluxes with the Godunov (or IT principle) ones analytically, we identify the cases that they are different. Nevertheless, with numerical experiments, we demonstrate that they are consistent in all the cases in the sense that the stationary states of both models are identical.

The remainder of this paper is organized as follows. In Section 2, we discuss the lane choice model based on Wardrop's user equilibrium principle and derive the fundamental diagram for the traffic system with priority vehicles and special lanes. In Section 3, we derive a new system of two coupled conservation laws. In Section 4, we present a new flux function for the Cell Transmission Model and analytically compare it with the IT principle. In Section 5, we compare the new flux function and the IT principle with numerical examples in terms of the stationary states and invariance property. In Section 6, we conclude the study with some future research directions.

2. Lane choice model and fundamental diagram

In this study we use a set of notations that are slightly different from those in Daganzo (1997):

- l: the proportion of the number of special lanes ($0 < l < 1$).
- $k(t, x)$: the total density of both priority and regular vehicles at time t and location x.
- $p(t, x)$: the proportion of the density of priority vehicles.
- $\xi(t, x)$: the proportion of the density on the special lanes.
- $v_1(t, x)$: the speed on the special lanes.
- $v_2(t, x)$: the speed on the regular lanes.

Hereafter we omit (t, x) unless necessary and assume that l is time- and location-independent. Without loss of generality, we also assume that p is positive. Therefore the density of the priority and regular vehicles are respectively pk and $(1 - p)k$, and the density on the special and regular lanes are respectively ξk and $(1 - \xi)k$. Clearly $\xi \leq p$, and the density of the priority vehicles that use the special and regular lanes are respectively ξk and $(p - \xi)k$. We define the 1- and 2-pipe regimes as follows.

Definition 2.1. *The traffic is in 1-pipe regime if and only if $\xi < p$; and 2-pipe regime if and only if $\xi = p$.*

Furthermore we assume the following speed- and flow-density relations: (i) $v = V(k)$: the non-increasing speed-density relation when all lanes, including both special and regular lanes, have the same density and speed; (ii) $q =$

$Q(k) = kV(k)$: the flow-density relation when all lanes have the same density and speed. Thus we have the speeds of special and regular lanes as follows.

$$v_1 = V\left(\frac{\xi}{l}k\right), \quad v_2 = V\left(\frac{1-\xi}{1-l}k\right) \tag{1}$$

The priority vehicles can use both special and regular lanes; they would choose the set of lanes that move faster. We thus assume the following Wardrop's user equilibrium principle for the lane choice behavior.

Definition 2.2 (Wardrop's user equilibrium principle)**.**

$$0 \le (v_1 - v_2) \perp (p - \xi) \ge 0 \tag{2}$$

where $a \perp b$ means $ab = 0$ for scalars a and b.

From Definitions 2.1 and 2.2, we have

Theorem 2.3. *Suppose that the speed-density relation is strictly decreasing. Then traffic is the 1-pipe regime, if and only if $\xi = l < p$, and in the 2-pipe regime, if and only if $\xi = p \le l$, that is*

$$\xi = \min\{p, l\}. \tag{3}$$

Proof. In the 1-pipe regime, we have $\xi < p$ and $v_1 = v_2$, and from (1) $V\left(\frac{\xi}{l}k\right) = V\left(\frac{1-\xi}{1-l}k\right)$ which leads to $\xi = l < p$. In the 2-pipe regime, we have $\xi = p$ and $v_2 \le v_1$, and from (1) $V\left(\frac{\xi}{l}k\right) \ge V\left(\frac{1-\xi}{1-l}k\right)$ which leads to $\frac{\xi}{l}k \le \frac{1-\xi}{1-l}k$, and $\xi = p \le l$. The reverse arguments can also be easily proved. □

Corollary 2.4. *The speeds of the priority and regular vehicles are v_1 and v_2 in Eq.(1), which can be written as*

$$v_1 = V\left(\frac{\min\{p,l\}}{l}k\right), \quad v_2 = V\left(\frac{1-\min\{p,l\}}{1-l}k\right) \tag{4}$$

Proof. It is straightforward to see that in both 1- and 2-pipe regimes, all priority vehicles have the same speed v_1, and all regular vehicles have the same speed v_2. Then from Theorem 2.3 we can have Eq.(4) from Eq.(1). □

Note that Theorem 2.3 is not true for the case of a triangular fundamental diagram because if both priority and regular vehicles travel at the same free-flow speed we cannot uniquely determine ξ from the information of p and l. Therefore, in the following, we assume that the priority vehicles only use the special lanes in the 2-pipe regime, which is exactly the same assumption made in Daganzo's theory (Daganzo, 1997), i.e., both Eqs. (3) and (4) hold.

3. The kinematic wave model

3.1. Conservation laws of two types of vehicles

Both the priority and regular vehicles satisfy the conservation law:

$$\frac{\partial pk}{\partial t} + \frac{\partial\{pk \cdot v_1\}}{\partial x} = 0, \quad \frac{\partial(1-p)k}{\partial t} + \frac{\partial\{(1-p)k \cdot v_2\}}{\partial x} = 0.$$

From the speed-density relationships in Eq.(4), we have the following kinematic wave models:

$$\frac{\partial pk}{\partial t} + \frac{\partial\left\{pk \cdot V\left(\frac{\min\{p,l\}}{l}k\right)\right\}}{\partial x} = 0, \quad \frac{\partial(1-p)k}{\partial t} + \frac{\partial\left\{(1-p)k \cdot V\left(\frac{1-\min\{p,l\}}{1-l}k\right)\right\}}{\partial x} = 0. \tag{5}$$

The two equations are well-defined with two unknown variables, k and p. Although they are not in the conservative form, we can derive new conservation laws by the following variable transformation.

We define two new variables, i.e., the densities on the special and regular lanes: $k_1 \equiv \frac{\min\{p,l\}}{l}k$, $k_2 \equiv \frac{1-\min\{p,l\}}{1-l}k$, where $k_1 \le k_2 \le \kappa$ implying that $k \le \kappa\frac{1-l}{1-\min\{p,l\}}$, and κ is the jam density. Further, since $pl = \max\{p, l\} \cdot \min\{p, l\}$, and $(1 - p) \cdot (1 - l) = (1 - \max\{p, l\}) \cdot (1 - \min\{p, l\})$, the density of the priority and regular vehicles can be expressed as

$$pk = y_1 k_1, \quad (1-p)k = y_2 k_2, \tag{6}$$

where $y_1 \equiv \max\{p, l\}$, $y_2 \equiv 1 - \max\{p, l\} = 1 - y_1$ that are interpreted as the space or lane proportion that is occupied by the priority [regular] vehicles. Hence $y_1 = p$ in the 1-pipe regime, and $y_1 = l$ in the 2-pipe regime. That is, the density proportion of the priority vehicles is irrelevant for y_1 in the 2-pipe regime. Finally, Eq.(5) can be re-written as

$$\frac{\partial}{\partial t} y_1 k_1 + \frac{\partial}{\partial x} y_1 k_1 V(k_1) = 0, \quad \frac{\partial}{\partial t} y_2 k_2 + \frac{\partial}{\partial x} y_2 k_2 V(k_2) = 0. \tag{7}$$

3.2. Two auxiliary conservation laws

Now we can obtain two new "lane-based" conservation laws in the following theorem.

Theorem 3.1. *Both k_1 and k_2 are conserved and satisfy the following LWR models:*

$$\frac{\partial k_1}{\partial t} + \frac{\partial Q(k_1)}{\partial x} = 0, \quad \frac{\partial k_2}{\partial t} + \frac{\partial Q(k_2)}{\partial x} = 0. \tag{8}$$

Proof. In the 1-pipe regime when $p > l$, $k_1 = k_2 = k$, and Eq.(7) leads to

$$\tfrac{\partial}{\partial t} pk + \tfrac{\partial}{\partial x} pQ(k) = 0, \quad \tfrac{\partial}{\partial t}(1-p)k + \tfrac{\partial}{\partial x}(1-p)Q(k) = 0.$$

Adding the two equation together, we have Eq.(8). In the 2-pipe regime when $p \le l$, Eq.(7) can be re-written as

$$l\left(\tfrac{\partial k_1}{\partial t} + \tfrac{\partial Q(k_1)}{\partial x}\right) = 0, \quad (1-l)\left(\tfrac{\partial k_2}{\partial t} + \tfrac{\partial Q(k_2)}{\partial x}\right) = 0,$$

which are equivalent to the two equations in Eq.(8) because $0 < l < 1$. □

Corollary 3.2. *For $k > 0$, y_1 travels forward with the priority vehicles, and y_2 travels forward with the regular vehicles. But y_1 and y_2 can travel at different speeds.*

Proof. Combining Eqs.(7) and (8), we have

$$k_1\left(\tfrac{\partial}{\partial t} y_1 + V(k_1)\tfrac{\partial}{\partial x} y_1\right) = 0, \quad k_2\left(\tfrac{\partial}{\partial t} y_2 + V(k_2)\tfrac{\partial}{\partial x} y_2 = 0\right),$$

Thus, for $k > 0$, along the priority vehicles' trajectories, $\frac{\mathrm{d}x}{\mathrm{d}t} = V(k_1)$, we have $\frac{\mathrm{d}}{\mathrm{d}t} y_1 = \frac{\partial}{\partial t} y_1 + \frac{\partial}{\partial x} y_1 \frac{\mathrm{d}x}{\mathrm{d}t} = 0$. That is, y_1 remains constant along the trajectory of a priority vehicle; in other words, it travels forward along priority vehicles. Similarly we can show that y_2 remains constant along regular vehicles' trajectories and travels forward with the regular vehicles. □

Therefore the two conservation laws in Eq.(8) are coupled together, as y_1 and y_2 cannot propagate forward independently. Otherwise, $y_1 + y_2$ may not equal to 1.

4. Cell Transmission Models

4.1. A new Cell Transmission Model

Let us describe a new Cell Transmission Model based on the discussion in the previous section. For cell i at time step j, we denote the total density and the priority vehicles' density proportion by k_i^j and p_i^j, respectively. Then, $\xi_i^j = \min\{p_i^j, l\}$, $y_{1,i}^j = \max\{p_i^j, l\}$, $y_{2,i}^j = 1 - y_{1,i}^j$, $k_{1,i}^j = \frac{\xi_i^j}{l} k_i^j$, $k_{2,i}^j = \frac{1-\xi_i^j}{1-l} k_i^j$. Discretize Eq.(7) in time and space, we first have the conservation of priority and regular vehicles:

$$\begin{aligned} y_{1,i}^{j+1} k_{1,i}^{j+1} &= y_{1,i}^j k_{1,i}^j + \tfrac{\Delta t}{\Delta x}(y_{1,i-1}^j q_{1,i-1}^j - y_{1,i}^j q_{1,i}^j), \\ y_{2,i}^{j+1} k_{2,i}^{j+1} &= y_{2,i}^j k_{2,i}^j + \tfrac{\Delta t}{\Delta x}(y_{2,i-1}^j q_{2,i-1}^j - y_{2,i}^j q_{2,i}^j), \end{aligned}$$

from which we can uniquely solve for k_i^{j+1} and p_i^{j+1}. Note that $\Delta x \le u \Delta t$ (CFL condition) must be satisfied. The total density and the proportion of the density of priority vehicles are updated through the following equations:

$$k_i^{j+1} = y_{1,i}^{j+1} k_{1,i}^{j+1} + y_{2,i}^{j+1} k_{2,i}^{j+1}, \qquad p_i^{j+1} = (y_{1,i}^{j+1} k_{1,i}^{j+1})/k_i^{j+1}. \tag{9a}$$

We next consider flux functions that determine the value of $q^j_{1,i-1}$ and $q^j_{2,i-1}$. To do so, we define the demand and supply functions as follows.

$$D(k) = Q(\min\{k, \kappa_c\}), \quad S(k) = Q(\max\{k, \kappa_c\})$$

where κ_c is the critical density. Thus, $q^j_{1,i-1} = \min\{D(k^j_{1,i-1}), S(k^j_{1,i})\}$, $q^j_{2,i-1} = \min\{D(k^j_{2,i-1}), S(k^j_{2,i})\}$ (Daganzo, 1994). This means that the proposed flux functions of priority and regular vehicles are given by

$$q^j_{p,i-1} = y^j_{1,i-1} q^j_{1,i-1} = y^j_{1,i-1} \min\{D(k^j_{1,i-1}), S(k^j_{1,i})\}, \tag{10a}$$

$$q^j_{r,i-1} = y^j_{2,i-1} q^j_{2,i-1} = y^j_{2,i-1} \min\{D(k^j_{2,i-1}), S(k^j_{2,i})\}. \tag{10b}$$

These are new and different from the IT principle (Daganzo et al., 1997).

The ideas behind the proposed flux functions (10) are as follows. As shown in Section 3.2, the original kinematic wave models may be decoupled into two independent LWR models if $y_1 + y_2 = 1$; to ensure $y_1 + y_2 = 1$, we use the "upstream" lane proportions that occupied by priority and regular vehicles because these proportions travel forward with vehicles unless the shock occurs. However, Eq.(10) does not capture the fact that y_1 and y_2 can travel at different speeds and thus is an approximation of the shock condition (see Daganzo, 1997). Nevertheless, the proposed CTM accurately captures the shock wave in the exact Riemann solution (or IT principle) as shown in later.

4.2. Analytical comparison with the Godunov flux

We analytically compare the proposed fluxes with the Godunov ones assuming a triangular fundamental diagram. In this setting, Daganzo (1997) derived Godunov fluxes analytically and proved that it was equivalent to the IT principle.

Let us consider the Riemann problem under the following jump conditions:

$$(k(0,x), p(0,x)) = \begin{cases} (k_L, p_L) & x < 0 \\ (k_R, p_R) & x > 0 \end{cases}.$$

Other variables can be calculated like the proceeding subsection. Then the proposed approximate flux is written as

$$q_p = y_{1,L} \min\{D(k_{1,L}), S(k_{1,R})\}, \quad q_r = y_{2,L} \min\{D(k_{2,L}), S(k_{2,R})\}. \tag{11}$$

Meanwhile, the IT flux (i.e., Godunov flux) can be expressed as

$$q^{IT}_p = \min\left\{d_{p,L},\ s_R \max\left\{\frac{s_{p,R}}{s_R}, \frac{d_{p,L}}{d_L}\right\}\right\}, \quad q^{IT}_r = \min\left\{d_{r,L},\ s_R \min\left\{\frac{s_{r,R}}{s_R}, \frac{d_{r,L}}{d_L}\right\}\right\}. \tag{12}$$

where the demands and supplies of priority, regular and all vehicles are $d_{p,L} = y_{1,L} D(k_{1,L})$, $d_{r,L} = y_{2,L} D(k_{2,L})$, $s_{p,R} = l S(k_{1,R})$, $s_{r,R} = (1-l) S(k_{2,R})$, $d_L = d_{p,L} + d_{r,L}$, $s_R = s_{p,R} + s_{r,R}$. This is a new concise representation of the IT principle.

From these, we can see that the proposed flux function (11) is much simpler than the IT principle (12). More importantly, it is a lane-based formulation and decoupuled. Note that both functions are equivalent when both upstream and downstream conditions are in the same regime because the KW models (5) and (7) reduce to the LWR models.

We next consider the case that the upstream traffic is in the 2-pipe regime and the downstream traffic is in the 1-pipe regime. Then the proposed flux function (11) reduces to

$$q_p = l \min\{D(k_{1,L}), S(k_{1,R})\}, \quad q_r = (1-l) \min\{D(k_{2,L}), S(k_{2,R})\},$$

while the IT principle (12) reduces to

$$q^{IT}_p = \min\left\{d_{p,L}, s_{p,R}\right\} = l \min\{D(k_{1,L}), S(k_{1,R})\}, \quad q^{IT}_r = \min\{d_{r,L}, s_{r,R}\} = (1-l) \min\{D(k_{2,L}), S(k_{2,R})\}.$$

Thus these are equivalent. The intuitive reason of this may be as follows. As Daganzo (1997) discussed, the solution of this case is a "lead vehicle type" where the upstream traffic follows the last vehicle of the downstream traffic. Therefore the upstream $y_{1,L}$ and $y_{2,L}$ would travel through the boundary $x = 0$ along with the upstream traffic.

The last and the most complicated case is that the upstream traffic is in the 1-pipe regime and the downstream traffic is in the 2-pipe regime. In this case, Eq.(11) is

$$q_p = p_L \min\{D(k_{1,L}), S(k_{1,R})\}, \quad q_r = (1-p_L) \min\{D(k_{2,L}), S(k_{2,R})\}.$$

Table 1. Scenarios and results (triangular fundamental diagram case)

No.	Case	Upstream	Downstream	Demand/Supply	Stationary State	Invariance
1	Subcase 1-A	congested	congested	$d_{p,L}/d_{r,L} \geq s_{p,R}/s_{r,R}$	✓	
2	Subcase 1-A	congested	congested	$d_{p,L}/d_{r,L} < s_{p,R}/s_{r,R}$	✓	
3	Subcase 1-A	congested	semi-congested	$d_{p,L}/d_{r,L} \geq s_{p,R}/s_{r,R}$	✓	
4	Subcase 1-A	congested	semi-congested	$d_{p,L}/d_{r,L} < s_{p,R}/s_{r,R}$	✓	
5	Subcase 1-A	free	congested	$d_{p,L}/d_{r,L} \geq s_{p,R}/s_{r,R}$	✓	
6	Subcase 1-A	free	congested	$d_{p,L}/d_{r,L} < s_{p,R}/s_{r,R}$	✓	
7	Subcase 1-A	free	semi-congested	$d_{p,L}/d_{r,L} \geq s_{p,R}/s_{r,R}$	✓	
8	Subcase 1-A	free	semi-congested	$d_{p,L}/d_{r,L} < s_{p,R}/s_{r,R}$	✓	
9	Subcase 1-B	free	congested	$d_{p,L}/d_{r,L} \geq s_{p,R}/s_{r,R}$	✓	✓
10	Subcase 1-B	free	semi-congested	$d_{p,L}/d_{r,L} \geq s_{p,R}/s_{r,R}$	✓	✓
11	Subcase 2	free	congested	$d_{p,L}/d_{r,L} \geq s_{p,R}/s_{r,R}$	✓	✓
12	Subcase 2	free	congested	$d_{p,L}/d_{r,L} < s_{p,R}/s_{r,R}$	✓	
13	Subcase 2	free	semi-congested	$d_{p,L}/d_{r,L} \geq s_{p,R}/s_{r,R}$	✓	✓
14	Subcase 2	free	semi-congested	$d_{p,L}/d_{r,L} < s_{p,R}/s_{r,R}$	✓	

However Eq.(12) cannot be specified further without other information, which implies that they are different. For example, if we consider $d_{p,L}/d_{r,L} < s_{p,R}/s_{p,R}$, the IT principle (12) reduces to

$$q_p^{IT} = \min\left\{d_{p,L}, s_{p,R}\right\} = \min\{p_L D(k_{1,L}), lS(k_{1,R})\}, \quad q_r^{IT} = \min\{d_{r,L}, s_{r,R}\} = \min\{(1-p_L)D(k_{2,L}), (1-l)S(k_{2,R})\}.$$

Thus we see that the proposed and the IT fluxes are different if the total demand exceeds the total supply.

5. Numerical experiments

5.1. Comparison with the Incremental Transfer principle

Assume a triangular fundamental diagram with free flow speed u, backward wave speed w and jam density κ. We numerically solve the Riemann problem for the transition from the 1-pipe to the 2-pipe regime by both CTMs with the proposed and IT fluxes and then compare the waves in the Riemann solution.

According to Daganzo et al. (1997), we divide the transition into 4 subcases:

- **Subcase 1-A**: the downstream traffic is in congested or semi-congested state, the upstream traffic is either in congested or in uncongested "subcase 1" (in the terminology in Daganzo, 1997), and the shock speed is negative.
- **Subcase 1-B**: the downstream traffic is in congested or semi-congested state, the upstream traffic is in uncongested "subcase 1", and the shock speed is positive.
- **Subcase 2**: the downstream traffic is in congested or semi-congested state and the upstream traffic is in uncongested "subcase 2".
- **Subcase 3**: the downstream traffic is in uncongested state.

where the semi-congested state is the situation where only regular lanes are congested; the "subcase 1" is the situation where the proportion of priority vehicles in upstream traffic is rich; the "subcase 2" is the situation where the proportion of priority vehicles in upstream traffic is so low. Note that we can easily prove that the proposed and IT fluxes are equivalent in Subcase 3 and thus we omit it in the numerical experiments.

Finally, 14 scenarios are enumerated in Table 1. In addition to the above subcases, they are further distinguished by the combinations of the upstream states, downstream states and the relations between the demand and supply proportions (i.e., $d_{p,L}/d_{r,L} \geq s_{p,R}/s_{r,R}$ or $d_{p,L}/d_{r,L} < s_{p,R}/s_{r,R}$). Other common settings are $l = 0.25$, $u = 100$ [km/h], $w = 20$ [km/h], $\kappa = 140$ [veh/km/lane], $\Delta x = 5$ [m], and $\Delta t = \Delta x/u = 0.18$ [sec].

Figure 1 shows Riemann solutions (i.e., density evolutions in time-space) of the IT principle and new CTM for Scenario 5 (Subcase 1-A). In Scenarios 5, the traffic evolves in the following order from the upstream to downstream: the 1-pipe free flow, 1-pipe congested, 2-pipe congested, and 2-pipe semi-congested states. From this figure, we see that the waves in the solutions of the new CTM accurately reproduce those of the IT principle.

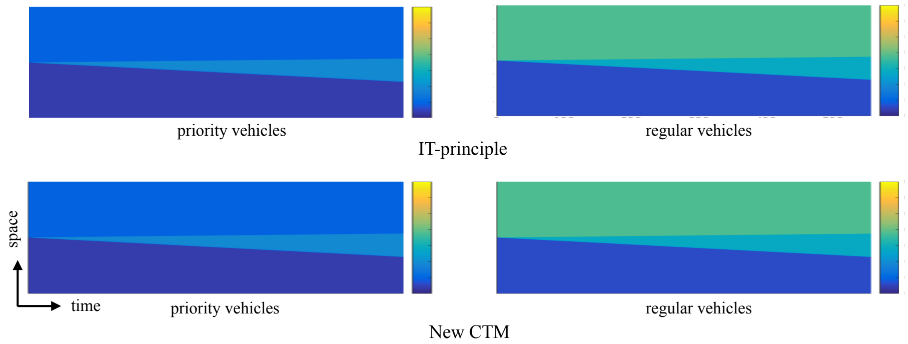

Fig. 1. Density plots with the IT principle and new CTM

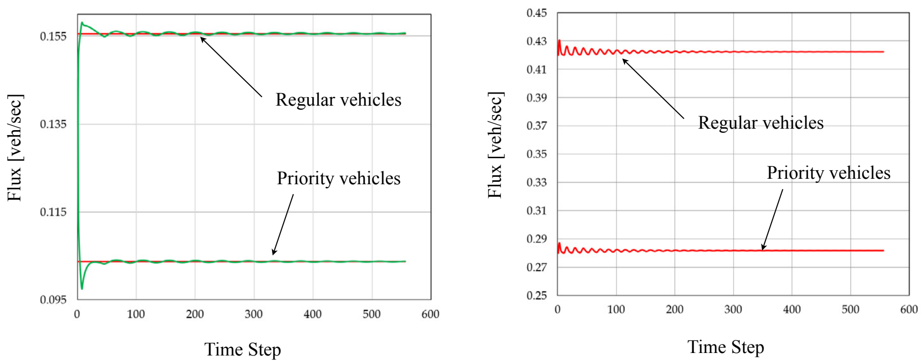

Fig. 2. Flux plots of the IT principle (red line) and the new CTM (green line). Left: triangular FD; right: Greenshields FD

For a more quantitative comparison, we define the stationary state, which finally covers all road section, as a state at a large time step I that satisfies the following criteria: $|(q^{j*}_{1,I} - q^{j*}_{1,I-1})/q^{j*}_{1,I-1}| \leq \epsilon$ and $|(q^{j*}_{2,I} - q^{j*}_{2,I-1})/q^{j*}_{2,I-1}| \leq \epsilon$, where $x_{j*} = (j*)\Delta x = 0$ and we set $\epsilon = 10^{-3}$ and $I = 500$. The result is shown in column "Stationary State" in Table 1; "✓" indicates that the traffic of both models reaches the stationary state and matches. This result shows that the new CTM and IT principle are consistent in all the cases in the sense that the stationary state is identical even though the analytical forms of the flux functions are different.

5.2. Invariance test

As well known, the (exact) Riemann solution must be self-similar, which means that "*a picture of the solution at any time t also represents the solution at another time, βt, if the space dimension in the picture is resealed by the factor β*" (Daganzo, 1997). This property in terms of the boundary flux is also known as "invariance principle" (Lebacque and Khoshyaran, 2005). In this section, we investigate the invariance property of the new CTM and IT principle.

Left figure in Figure 2 shows that the fluxes of priority and regular vehicles at $x_{j*} = 0$ for Scenario 5; red lines show the IT fluxes and green lines show the proposed fluxes. We see that the proposed fluxes converge to the stationary state with oscillation while the IT fluxes are stationary. This is almost the case, i.e., the IT fluxes are invariant for all the scenarios and the proposed fluxes are invariant for only scenarios where the shock moves forward summarized in column "Invariance" in Table 1. This difference apparently comes from the difference in the analytical flux function. Also, the reason why the new CTM is non-invariant would be because the flux function includes a demand-proportional supply distribution rule (see Jin and Zhang, 2003, for the other example) as pointed out in Lebacque and Khoshyaran (2005).

On the other hand, although the IT principle can be invariant (i.e., it is equivalent to the Godunov flux) for triangular fundamental diagrams, the invariance property cannot be universally expected to the IT principle because it also includes a demand-proportional supply distribution rule. To demonstrate this concretely, we show an oscillatory solution of the IT principle with a Greenshields fundamental diagram in right figure in Figure 2.

From the results, we consequently conclude that (i) both the IT principle and new CTM generally are non-invariant; (ii) both schemes have the same stationary states. These imply that the invariant (or continuous) flux function associ-

ated with these discrete or numerical flux functions is identical; it could be derived by using the new Riemann solver (Jin et al., 2009; Jin, 2017a) that can incorporate a discrete flux function as an entropy condition. Since the proposed flux function is much simpler than the IT principle, it would make us examine the invariant flux function form easier.

6. Conclusion

In this study we first derived the fundamental diagram of a traffic system with priority vehicles and special lanes based on Wardrop's user equilibrium principle. By transforming the variables we introduced two simple auxiliary conservation laws, from which we defined traffic demand and supply functions as in the traditional CTM. Then we presented a new boundary flux function and CTM to solve the kinematic wave model of the traffic system. With both analytical and numerical studies we found that (i) the new flux function is consistent with the IT principle in most of the cases; (ii) under all conditions, the new flux function and the IT principle lead to the same stationary states; and (iii) the new flux function is not invariant for either triangular or Greenshields fundamental diagrams, and the IT principle is not invariant for the Greenshields fundamental diagram. These results suggest that the analytical method and solutions for the Riemann problem in Daganzo (1997) may not be applicable for general fundamental diagrams; but the IT principle and the new junction flux function are still applicable.

A Riemann solver for general fundamental diagrams is still elusive for the kinematic wave model of such a traffic system. Even though traffic in the 1-pipe regime is both unifiable and FIFO (first-in-first-out), that in the 2-pipe regime can be neither unifiable nor FIFO (Jin, 2017b). Such a system is much more complicated than unifiable or FIFO ones; the latter are already more challenging than both the traditional LWR model, which is both unifiable and FIFO. In the future we will be interested in developing such a Riemann solver.

In the future, we will be interested in studying traffic systems with priority vehicles and special high-occupancy-toll (HOT) lanes, such that regular vehicles can use HOT subject to tolls. The introduction of two auxiliary conservation laws leads to simpler formulations of both the IT principle and the new flux function. In particular we will be interested in developing the Cell Transmission Model for the more complex system.

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References

Daganzo, C.F., 1994. The cell transmission model: A dynamic representation of highway traffic consistent with the hydrodynamic theory. Transportation Research Part B 28, 269–287.
Daganzo, C.F., 1995. The cell transmission model, part II: Network traffic. Transportation Research Part B 29, 79–93.
Daganzo, C.F., 1997. A continuum theory of traffic dynamics for freeways with special lanes. Transportation Research Part B 31, 83–102.
Daganzo, C.F., Lin, W.H., Del Castillo, J.M., 1997. A simple physical principle for the simulation of freeways with special lanes and priority vehicles. Transportation Research Part B 31, 103–125.
Jin, W.L., 2017a. A Riemann solver for a system of hyperbolic conservation laws at a general road junction. Transportation Research Part B 98, 21–41.
Jin, W.L., 2017b. Unifiable multi-commodity kinematic wave model. Transportation Research Procedia: Proceedings of the 22nd International Symposium on Transportation and Traffic Theory 23, 137–156.
Jin, W.l., Chen, L., Puckett, E., 2009. Supply-demand diagrams and a new framework for analyzing the inhomogeneous Lighthill-Whitham-Richards model, in: Lam, W.H.K., Wong, S.C., Lo, H.K. (Eds.), Proceedings of the 18th International Symposium on Transportation and Traffic Theory, Springer US, Boston, MA. pp. 603–635.
Jin, W.L., Zhang, H.M., 2003. On the distribution schemes for determining flows through a merge. Transportation Research Part B 37, 521–540.
Lebacque, J.P., 1996. The Godunov scheme and what it means for first order traffic flow models, in: Lesort, J.B. (Ed.), Proceedings of the 13th International Symposium on Transportation and Traffic Theory, Pergamon-Elsevier, Lyon. pp. 647–677.
Lebacque, J.P., Khoshyaran, M.M., 2005. First–order macroscopic traffic flow models: Intersection modeling, network modeling, in: Mahmassani, H.S. (Ed.), Proceedings of the 16th International Symposium on Transportation and Traffic Theory, Elsevier, Maryland. pp. 365–386.
Lighthill, M.J., Whitham, G.B., 1955. On kinematic waves. II. A theory of traffic flow on long crowded roads. Proceedings of the Royal Society A 229, 317–345.
Richards, P.I., 1956. Shock waves on the highway. Operations Research 4, 42–51.