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Journal of Functional Analysis 241 (2006) 232–267
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

Invariant differential operators and the range of the matrix Radon transform

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Received 1 February 2006; accepted 12 April 2006

Available online 8 June 2006

Communicated by Richard B. Melrose

Abstract

In this paper we characterize the range of the matrix Radon transform by invariant differential operators. This generalizes analogous results for the d-plane transform in $\mathbb{R}^n$.

Keywords: Matrix Radon transform; Invariant differential operators

1. Introduction

The matrix Radon transform can be viewed as a generalization of the Radon d-plane transform in $\mathbb{R}^n$. Various aspects of this transform, which was introduced by Petrov [14] in 1967, have been investigated by several authors.

The inversion of this transform on various function spaces has been fairly well studied. (See Petrov's papers [14,15], in addition to [12,17–20,22].) The detailed and comprehensive monograph by Ournycheva and Rubin [11] presents several new inversion formulas, making use, of, among other things, Gårding–Gindikin fractional integrals or shifted dual transforms.

In this paper, we investigate the range of this transform. When the space Ξ of matrix planes has the same dimension as the source manifold $M_{n,k}$ of $n \times k$ real matrices, this range can be characterized by moment and/or Cavalieri conditions. (See Petrov's paper [16].) In particular,

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0022-1236/$ – see front matter
doi:10.1016/j.jfa.2006.04.008

when $k = 1$, Petrov's conditions reduce to the usual moment conditions characterizing the range of the classical Radon transform on $\mathcal{S}(\mathbb{R}^n)$ or $\mathcal{D}(\mathbb{R}^n)$.

Our objective is to characterize the range when $\dim(\Xi) > \dim(M_{n,k})$. When $k = 1$, this transform is just the usual d-plane transform on $\mathbb{R}^n$, where $d = n - m$ and $m > 1$. In this case the range can be characterized by differential equations, either in parametric form (i.e. using John's equations [2,8]) or group-invariant form [4,21]. Our main result, Theorem 6.2, is a range characterization of the matrix Radon transform on the Schwartz space $\mathcal{S}(M_{n,k})$, for arbitrary k, using differential equations of the latter variety. (This has the advantage of allowing us to use various Lie-theoretic tools.) Along the way, in Theorem 6.7, we will prove a weak Nullstellensatz for real irreducible polynomials in $\mathbb{R}^n$ which we hope might be of independent interest.

Our range-characterizing differential operators turn out to arise from Pfaffian-type elements of the universal enveloping algebra of the Lie group $(\mathrm{O}(n) \times \mathrm{O}(k)) \rtimes M_{n,k}$. If we consider this group to be a Cartan motion group of the orthogonal group $\mathrm{O}(n + k)$, then our operators turn out to be "contractions" of appropriate invariant elements from the latter group.

2. The space of matrix planes

Let $M_{n,k}$ denote the vector space of real $n \times k$ matrices. In what follows, we assume that $k \leqslant n$. We define the *rank* of $M_{n,k}$ to be k. A *Stiefel matrix* is a real matrix whose columns are orthonormal. Let $\mathrm{St}(n, m)$ denote the set of $n \times m$ Stiefel matrices, identified in the natural way with the Stiefel manifold of orthonormal m-frames in $\mathbb{R}^n$. (Here, of course, $m \leqslant n$.)

Fix a Stiefel matrix $F \in \mathrm{St}(n, m)$ and an $m \times k$ real matrix B. Define the subset $\xi[F, B]$ of $M_{n,k}$ by

$$\xi[F, B] = \{X \in M_{n,k} \mid {}^tFX = B\}. \tag{2.1}$$

It is clear that $\xi[F, B]$ is an affine plane in $M_{n,k}$ of dimension $(n - m)k$. In particular, when $k = 1$, $\xi[F, B]$ is an $(n - m)$-dimensional plane in $M_{n,1} = \mathbb{R}^n$. Following the terminology in [15], we call $\xi[F, B]$ a *matrix $(n - m)$-plane*, or, more simply, a *matrix plane*.

Note that $\xi[F, B] = FB + \xi[F, 0]$, and from this it is not hard to see that $\xi[F, B] = \xi[F', B']$ iff $F' = F\sigma$ and $B' = \sigma^{-1}B$ for some $\sigma \in \mathrm{O}(m)$. Hence the space Ξ of matrix $(n - m)$-planes can be identified with the (total space of) the associated bundle $\mathrm{St}(n, m) \times_{\mathrm{O}(m)} M_{m,k}$ over the Grassmannian $G_{n,m}$ of m-dimensional subspaces of $\mathbb{R}^n$.

It follows that Ξ has dimension $m(n - m + k)$; hence $\dim M_{n,k} \leqslant \dim \Xi$ iff $k \leqslant m \leqslant n$. In order to avoid considering the trivial cases in the Radon transforms we investigate in this paper, we will assume throughout that $k \leqslant m < n$.

$M_{n,k}$ is, of course, just a Euclidean space, with inner product $\langle X, Y\rangle = \mathrm{tr}({}^tXY)$. We provide it and its matrix planes with the usual Euclidean structures; the function spaces $\mathcal{D}(M_{n,k}), \mathcal{S}(M_{n,k}), L^2(M_{n,k})$, etc., then have their usual meaning.

The compact Lie group $\mathrm{O}(n) \times \mathrm{O}(k)$ acts naturally on $M_{n,k}$ via left and right multiplication: $(u, v) \cdot X = uXv^{-1}$. We define the *matrix motion group G* of $M_{n,k}$ to be the semidirect product $(\mathrm{O}(n) \times \mathrm{O}(k)) \rtimes M_{n,k}$, where the group operation on $M_{n,k}$ is just addition. Denote a typical element of G by $(u, v; E)$, where $u \in \mathrm{O}(n)$, $v \in \mathrm{O}(k)$, and $E \in M_{n,k}$. The multiplication in G is given by

$$(u_1, v_1; E_1) \cdot (u_2, v_2; E_2) = (u_1u_2, v_1v_2; E_1 + u_1E_2v_1^{-1}) \tag{2.2}$$

and inversion by

$$(u, v; E)^{-1} = \bigl(u^{-1}, v^{-1}; -u^{-1}Ev\bigr). \tag{2.3}$$

G acts transitively on $M_{n,k}$ via $(u, v; E) \cdot X = uXv^{-1} + E$; when $k > 1$, it is a proper subgroup of the group of all rigid motions of the Euclidean space $M_{n,k}$. If we identify $M_{n,k}$ with the tangent space of the real Grassmannian $G_{n+k,n} = \mathrm{O}(n+k)/(\mathrm{O}(n) \times \mathrm{O}(k))$ at the identity coset o, we see that G is just the Cartan motion group of the symmetric space $G_{n+k,n}$ at o.

The motion group G permutes the matrix planes in Ξ via $g \cdot \xi = \{g \cdot X \mid X \in \xi\}$, for $g \in G$ and $\xi \in \Xi$. Explicitly, one can see that this action is given by

$$\begin{aligned}(u, v; E) \cdot \xi[F, B] &= \bigl\{X \in M_{n,k} \mid (u, v; E)^{-1} \cdot X \in \xi[F, B]\bigr\} \\ &= \xi\bigl[uF, Bv^{-1} + {}^tFu^{-1}E\bigr]. \end{aligned} \tag{2.4}$$

It is clear from (2.4) and the differentiable structure of $\Xi = \mathrm{St}(n, m) \times_{\mathrm{O}(m)} M_{m,k}$ that G acts smoothly and transitively on Ξ, preserving its fibers.

Conventions. If Φ is any mapping on Ξ, we will denote the image $\Phi(\xi[F, B])$ by $\Phi[F, B]$ instead, for simplicity. In addition, by "differential operator" on a manifold we will mean a smooth linear differential operator on the manifold.

3. The matrix Radon transform

Following [14] we define the *matrix Radon transform* R on functions on $M_{n,k}$ by

$$Rf[F, B] = \int\limits_{\xi[F,B]} f(X)\, dm(X), \tag{3.1}$$

for all suitable functions f on $M_{n,k}$, where dm is the Euclidean measure on $\xi[F, B]$. (We will work primarily with f in the Schwartz class $\mathcal{S}(M_{n,k})$.) R is then the Radon transform associated with the double fibration

$$\begin{array}{ccc} & G/(K \cap H) & \\ \swarrow & & \searrow \\ M_{n,k} = G/K & & \Xi = G/H \end{array} \tag{3.2}$$

(See [7] for the general theory of homogeneous spaces in duality and their associated integral transforms.) Here K and H are the isotropy subgroups of the G-action on $M_{n,k}$ and Ξ, respectively, at suitable points. To fix matters, we set $K = \mathrm{O}(n) \times \mathrm{O}(k)$, the subgroup of G fixing $0 \in M_{n,k}$, and we let H be the subgroup of G fixing the matrix plane $\xi_0 = \xi[F_0, 0]$, where F_0 is the m-frame $[e_1, \ldots, e_m]$. (Here e_j denotes the jth basis vector of $\mathbb{R}^n$.) Clearly, ξ_0 is the vector subspace of $M_{n,k}$ consisting of the $n \times k$ matrices of the form

$$\begin{pmatrix} 0_{m\times k} \\ --- \\ * \end{pmatrix}$$

and from this it is easy to see that $H = ([\mathrm{O}(m) \times \mathrm{O}(n-m)] \times \mathrm{O}(k)) \rtimes \xi_0$. The incidence relation corresponding to the diagram (3.2) is just inclusion.

When $k = 1$, R is just the usual $(n-m)$-plane Radon transform on $\mathbb{R}^n$. The papers [11–15] deal with various aspects of the transform R. For example, [11] gives very nice inversion formulas for R on various function spaces in terms of Gårding–Gindikin fractional integrals or shifted dual transforms when $\dim M_{n,k} \leqslant \dim \Xi$; in [15] range and support theorems are given for R when $m = k$ (so that $\dim M_{n,k} = \dim \Xi$).

Since left multiplication by $\mathrm{O}(m)$ preserves the inner product $\langle B, C\rangle = \mathrm{tr}({}^tBC)$ on the vector space $M_{m,k}$, the Euclidean inner product on $M_{m,k}$ gives rise to a Riemannian structure on the associated bundle $\Xi = \mathrm{St}(n,m) \times_{\mathrm{O}(m)} M_{m,k}$. If $\xi = \xi[F, B]$, we put $|\xi| = \|B\|$; this represents the distance from $0 \in M_{n,k}$ to ξ. As in [14] we let $\mathcal{S}(\Xi)$ be the space of all C^∞ functions φ on Ξ which are rapidly decreasing on the fibers of Ξ. In the next section, we will give a more precise definition of $\mathcal{S}(\Xi)$ suitable for our purposes, and equivalent to that in [14]. (An equivalent definition is also presented further down in this paper, in Eqs. (6.50) and (6.51).)

Each fiber $\{\xi[F, B] \mid B \in M_{m,k}\}$ of the vector bundle $\Xi = \mathrm{St}(n,m) \times_{\mathrm{O}(m)} M_{m,k}$ corresponds to a *spread* (i.e., a parallel family) of matrix planes (see [2]), and every integral over $M_{n,k}$ decomposes into a double integral along this spread:

$$\int_{M_{n,k}} f(X)\,dX = \int_{B\in M_{m,k}} \int_{\xi[F,B]} f(X)\,dm(X)\,dB = \int_{B\in M_{m,k}} Rf[F,B]\,dB, \tag{3.3}$$

for any $f \in L^1(M_{n,k})$. (3.3) is easy to verify when $F = F_0$; it holds in general for any fiber since left multiplication by each $u \in \mathrm{O}(n)$ is an isometry on $M_{n,k}$ and the $\mathrm{O}(n)$-action is transitive on the spreads.

Let $f \mapsto \widetilde{f}$ denote the Fourier transform on $M_{n,k}$:

$$\widetilde{f}(Y) = \int_{M_{n,k}} f(X) e^{-i\,\mathrm{tr}({}^tYX)}\,dX, \quad f \in \mathcal{S}(M_{n,k}), \tag{3.4}$$

and let $\varphi \mapsto \mathcal{F}\varphi$ denote the Fourier transform on the fibers of Ξ:

$$\mathcal{F}\varphi[F, C] = \int_{M_{m,k}} \varphi[F, B] e^{-i\,\mathrm{tr}({}^tBC)}\,dB, \quad \varphi \in \mathcal{S}(\Xi). \tag{3.5}$$

It is not hard to show (using Eq. (4.4)) that $\mathcal{F}\varphi$ is a well-defined function in $\mathcal{S}(\Xi)$; we call it the *partial Fourier transform* of φ.

Now the matrix Radon transform R maps $\mathcal{S}(M_{n,k})$ into $\mathcal{S}(\Xi)$ (see [15]), and we have the following *Fourier slice theorem*: if $f \in \mathcal{S}(M_{n,k})$ and $\xi[F, C] \in \Xi$, then by (3.3)

$$\begin{aligned}\widetilde{f}(FC) &= \int_{M_{n,k}} f(X) e^{-i\,\mathrm{tr}({}^tC\,{}^tFX)}\,dX = \int_{B\in M_{m,k}} \int_{\xi[F,B]} f(X)\,dm(X)\, e^{-i\,\mathrm{tr}({}^tCB)}\,dB \\ &= \int_{M_{m,k}} Rf[F, B] e^{-i\,\mathrm{tr}({}^tCB)}\,dB = \mathcal{F}(Rf)[F, C].\end{aligned} \tag{3.6}$$

Note that (3.6) also holds for any $f \in L^1(M_{n,k})$ since by (3.3), Rf is a.e. defined and integrable on each spread. As a consequence, R is injective on $L^1(M_{n,k})$.

4. The universal enveloping algebra of G and the matrix Radon transform

Let $\mathfrak{g}$ denote the Lie algebra of the matrix motion group G, and let $U(\mathfrak{g})$ be its universal enveloping algebra. The matrix Radon transform R commutes with the left action by G, $\mathfrak{g}$, and $U(\mathfrak{g})$ on functions on $M_{n,k}$ and Ξ. To be precise, let λ and ν denote the left regular representations of G on smooth functions on $M_{n,k}$ and Ξ, respectively:

$$\lambda(g)f(X) = f^g(X) = f\big(g^{-1}\cdot X\big), \quad g \in G,\ X \in M_{n,k}, \tag{4.1}$$

for all $f \in C^\infty(M_{n,k})$; and

$$\nu(g)\varphi(\xi) = \varphi^g(\xi) = f\big(g^{-1}\cdot \xi\big), \quad g \in G,\ \xi \in \Xi, \tag{4.2}$$

for all $\varphi \in C^\infty(\Xi)$. Then denote by $d\lambda$ and $d\nu$ the corresponding infinitesimal left regular representations of $U(\mathfrak{g})$ on $C^\infty(M_{n,k})$ and $C^\infty(\Xi)$, respectively. We have, in particular,

$$d\lambda(Y_1\cdots Y_r)f(X) = \left\{\frac{\partial^r}{\partial t_1\cdots\partial t_r} f\big(\exp(-t_rY_r)\cdots\exp(-t_1Y_1)\cdot X\big)\right\}_{t_1=0,\ldots,t_r=0} \tag{4.3}$$

for $Y_1,\ldots,Y_r \in \mathfrak{g}$. $d\lambda$ and $d\nu$ are algebra homomorphisms from $U(\mathfrak{g})$ into the algebra of differential operators on $M_{n,k}$ and Ξ, respectively.

Let $\mathfrak{z}(\mathfrak{g})$ denote the subalgebra of $U(\mathfrak{g})$ consisting of the elements invariant under $\mathrm{Ad}(G)$. Since $\lambda(g)\circ d\lambda(U)\circ\lambda(g^{-1}) = d\lambda(\mathrm{Ad}(g)U)$ for all $g \in G$ and all $U \in U(\mathfrak{g})$, $d\lambda(\mathfrak{z}(\mathfrak{g}))$ consists of G-invariant differential operators on $M_{n,k}$.

At this point let us provide a precise definition of the Schwartz space $\mathcal{S}(\Xi)$. We say that a C^∞ function φ on Ξ belongs to $\mathcal{S}(\Xi)$ iff for any $U \in U(\mathfrak{g})$ and for any $N \in \mathbb{Z}^+$, φ satisfies the condition

$$\|\varphi\|_{N,U} := \sup_{\xi\in\Xi} |\xi|^N \big|d\nu(U)\varphi(\xi)\big| < +\infty. \tag{4.4}$$

The vector space $\mathcal{S}(\Xi)$, equipped with the locally convex topology defined by the seminorms $\|\ \|_{N,U}$, is a Fréchet space.

Since the matrix motion group G is a Lie subgroup of the group of rigid motions on $M_{n,k}$, we find that $\lambda(g)$ and $d\lambda(U)$ leave $\mathcal{S}(M_{n,k})$ invariant, for any $g \in G$ and $U \in U(\mathfrak{g})$. Likewise, it is easy to see from (2.4) and that $\nu(g)$ leaves $\mathcal{S}(\Xi)$ invariant, and, certainly from (4.4), $d\nu(U)$ leaves $\mathcal{S}(\Xi)$ invariant.

It is, moreover, a straightforward (albeit tedious) computation, again using (2.4) and (4.4), to show that ν is a strongly continuous representation of G on $\mathcal{S}(\Xi)$. Finally, the partial Fourier transform $\varphi \mapsto \mathcal{F}\varphi$ can be shown to be a topological isomorphism of $\mathcal{S}(\Xi)$ onto itself.

From the double fibration (3.2) we have $R(\lambda(g)f) = \nu(g)Rf$ for all $f \in \mathcal{S}(M_{n,k})$. Then using (4.3) we obtain

$$R\big(d\lambda(U)f\big) = d\nu(U)\,Rf. \tag{4.5}$$

By the Fourier slice theorem (3.6), R is injective on $\mathcal{S}(M_{n,k})$. Hence by (4.5), $\ker(d\nu) \subset \ker(d\lambda)$. If $U \in \ker(d\lambda)$, (4.5) also shows that any function φ in the range $R\mathcal{S}(M_{n,k})$ satisfies the system of differential equation $d\nu(U)\varphi = 0$. These equations are nontrivial if $\ker(d\lambda) \setminus \ker(d\nu) \neq 0$.

In this paper we will show that when $k < m$, these equations are sufficient to characterize the range as well, and that in fact there exists a single element $W \in \mathfrak{z}(\mathfrak{g})$ of order $2k+2$ such that

$$R\mathcal{S}(M_{n,k}) = \{\varphi \in \mathcal{S}(\Xi) \mid d\nu(W)\varphi = 0\}. \tag{4.6}$$

5. A central element in $U(\mathfrak{g})$

For the rest of this paper, we assume that $k < m$. In this section, we will use the adjoint relations in $\mathfrak{g}$ to produce the element $W \in \mathfrak{z}(\mathfrak{g})$ asserted in Eq. (4.6). Now $\mathfrak{g} = (\mathrm{so}(n) \times \mathrm{so}(k)) \rtimes M_{n,k} \cong \mathrm{so}(n) \oplus \mathrm{so}(k) \oplus M_{n,k}$. For a basis of $\mathrm{so}(n)$ we take the elementary $n \times n$ skew-symmetric matrices X_{ij}, where $1 \leqslant i < j \leqslant n$; likewise we take for a basis of $\mathrm{so}(k)$ the elementary $k \times k$ skew-symmetric matrices Y_{rs}, where $1 \leqslant r < s \leqslant k$. Finally, we take for a basis of $M_{n,k}$ the elementary $n \times k$ matrices E_{pq}, with $1 \leqslant p \leqslant n$, $1 \leqslant q \leqslant k$. Then $\mathfrak{g}$ has basis $\{X_{ij}, Y_{rs}, E_{pq}\}$. Here we identify $X_{ij} \in \mathrm{so}(n)$ with $(X_{ij}, 0, 0) \in \mathfrak{g}$, etc. The group laws (2.2) and (2.3) for G immediately imply the following adjoint relations on $\mathfrak{g}$:

$$\mathrm{Ad}(u, 1; 0)(0, 0; E) = (0, 0; uE), \tag{5.1}$$

$$\mathrm{Ad}(1, v; 0)(0, 0; E) = (0, 0; Ev^{-1}), \tag{5.2}$$

for $u \in \mathrm{O}(n)$, $v \in \mathrm{O}(k)$, and $E \in M_{n,k}$. Likewise, we have

$$\mathrm{Ad}(1, 1; E)(X, 0; 0) = (X, 0; -XE), \tag{5.3}$$

$$\mathrm{Ad}(1, 1; E)(0, Y; 0) = (0, Y; EY) \tag{5.4}$$

for $X \in \mathrm{so}(n)$, $Y \in \mathrm{so}(k)$.

From (5.3) we see that

$$\mathrm{ad}(E_{rs})X_{ij} = \delta_{ir}E_{js} - \delta_{jr}E_{is}. \tag{5.5}$$

To facilitate what follows, we introduce some additional notation. For positive integers $d \leqslant r$, let $P_{d,r}$ denote the set of all permutations of $\{1, \ldots, r\}$ taken d at a time, and let $T_{d,r}$ denote the collection of all d-element subsets of $\{1, \ldots, r\}$, considered as *increasing* sequences $J = (j_1, \ldots, j_d)$ of length d in $\{1, \ldots, r\}$. Using this convention, we have $T_{d,r} \subset P_{d,r}$. For $J \in T_{d,r}$, let $\mathfrak{S}(J)$ denote the set of all permutations of J (written as ordered r-tuples), so that $P_{d,r} = \bigcup_{J \in T_{d,r}} \mathfrak{S}(J)$. If $J' \in \mathfrak{S}(J)$, we denote its sign by $\epsilon(J')$.

Let $A = (a_{ij})$ be any $r \times s$ matrix with entries in some commutative ring. If $I' = (i_1, \ldots, i_d)$ and $J' = (j_1, \ldots, j_d)$ belong to $P_{d,r}$ and $P_{d,s}$, respectively, we let $A_{I',J'}$ denote the $d \times d$ submatrix $(a_{i_k j_l})_{k,l=1}^{r}$.

If $r = s$, so that A is a square matrix, and $d < r$, then the Laplace expansion theorem states that

$$\det(A) = \sum_{I,J} \epsilon(I, J) \det(A_{(1,\ldots,d),I}) \det(A_{(d+1,\ldots,r),J}), \tag{5.6}$$

where the sum is taken over all $I \in T_{d,r}$, $J \in T_{r-d,r}$ such that $I \cap J = \emptyset$ (a "$(d, r-d)$-shuffle"). Here $\epsilon(I, J)$ denotes the sign of the shuffle (I, J) of $\{1, \ldots, r\}$.

Let $\mathbf{E}$ denote the $n \times k$ matrix whose (i, j)-entry is the vector $E_{ij} \in M_{n,k}$. Fix $d \leqslant k$. For $I \in T_{d,n}$ and $J \in T_{d,k}$, let $D_{I,J} = \det(\mathbf{E}_{I,J}) \in U(M_{n,k}) \subset U(\mathfrak{g})$. Explicitly, if $I = \{i_1, \ldots, i_d\}$ and $J = \{j_1, \ldots, j_d\}$, $D_{I,J}$ is the element of $U(\mathfrak{g})$ given by

$$D_{I,J} = \sum_{\sigma \in \mathfrak{S}(I)} \epsilon(\sigma) E_{\sigma(i_1) j_1} \cdots E_{\sigma(i_d) j_d}. \tag{5.7}$$

Note that all the factors in each term above commute.

Now from (5.1), we see that $\mathrm{Ad}(u, 1; 0) E_{ij} = \sum_{l=1}^{n} u_{li} E_{lj}$ for all $u \in \mathrm{O}(n)$. This can be generalized to the following result.

Lemma 5.1. *Let $I \in T_{d,n}$ and $J \in T_{d,k}$. Then for $u \in \mathrm{O}(n)$, we have*

$$\mathrm{Ad}(u, 1; 0) D_{I,J} = \sum_{L \in T_{d,n}} \det(u_{L,I}) D_{L,J}. \tag{5.8}$$

Proof. Let $I = \{i_1, \ldots, i_d\}$ and $J = \{j_1, \ldots, j_d\}$. Then

$$\begin{aligned}
\mathrm{Ad}(u, 1; 0) D_{I,J} &= \mathrm{Ad}(u, 1; 0) \sum_{\sigma \in \mathfrak{S}(I)} \epsilon(\sigma) E_{\sigma(i_1) j_1} \cdots E_{\sigma(i_d) j_d} \\
&= \sum_{\sigma \in \mathfrak{S}(I)} \epsilon(\sigma) \left(\sum_{l=1}^{n} u_{l \sigma(i_1)} E_{l j_1} \right) \cdots \left(\sum_{l=1}^{n} u_{l \sigma(i_d)} E_{l j_d} \right) \\
&= \sum_{\sigma \in \mathfrak{S}(I)} \epsilon(\sigma) \sum_{1 \leqslant l_1, \ldots, l_d \leqslant n} u_{l_1 \sigma(i_1)} \cdots u_{l_d \sigma(i_d)} E_{l_1 j_1} \cdots E_{l_d j_d} \\
&= \sum_{1 \leqslant l_1, \ldots, l_d \leqslant n} \det(u_{l_k i_r}) E_{l_1 j_1} \cdots E_{l_d j_d} \\
&= \sum_{\substack{l_1, \ldots, l_d \\ \text{different}}} \det(u_{l_k i_r}) E_{l_1 j_1} \cdots E_{l_d j_d} \\
&= \sum_{\substack{L \in T_{d,n} \\ L = \{l_1, \ldots, l_d\}}} \sum_{\tau \in \mathfrak{S}(L)} \epsilon(\tau) \det(u_{L,I}) E_{\tau(l_1) j_1} \cdots E_{\tau(l_d) j_d} \\
&= \sum_{L \in T_{d,n}} \det(u_{L,I}) D_{L,A}. \qquad \square
\end{aligned}$$

One can prove a similar result for the right $\mathrm{O}(k)$ action:

Lemma 5.2. *Let $I \in T_{d,n}$ and $J \in T_{d,k}$. Then for $v \in \mathrm{O}(k)$, we have*

$$\mathrm{Ad}(1, v; 0) D_{I,J} = \sum_{R \in T_{d,k}} \det(v_{R,J}) D_{I,R}. \tag{5.9}$$

Next let $I \in T_{k+2,n}$. (Recall that we are assuming that $k < m < n$.) Denote by V_I the element of $U(\mathfrak{g})$ given by the expression

$$V_I = \sum_{\substack{i<j \\ \{i,j\}\subset I}} \epsilon\big(\{i,j\}, I \setminus \{i,j\}\big) X_{ij} D_{I\setminus\{i,j\}}. \tag{5.10}$$

Here we have put $D_{I\setminus\{i,j\}} = D_{I\setminus\{i,j\},\{1,\ldots,k\}}$, for simplicity. Note that each term $X_{ij} D_{I\setminus\{i,j\}}$ in the sum above is an order $k+1$ product in $U(\mathfrak{g})$ in which all the factors commute.

As an example, when $k=1$ and $I \in T_{3,n}$, $I\colon i<j<l$, we have

$$V_I = X_{ij}E_l - X_{il}E_j + X_{jl}E_i.$$

We can also view V_I as follows. Let $\mathbf{X}$ denote the $n \times n$ skew-symmetric matrix with vector entries X_{ij}. That is, we put

$$\mathbf{X} = \begin{pmatrix} 0 & X_{12} & \ldots & X_{1n} \\ -X_{12} & 0 & \ldots & X_{2n} \\ \ldots & \ldots & \ddots & \ldots \\ -X_{1n} & -X_{2n} & \ldots & 0 \end{pmatrix}. \tag{5.11}$$

If $\mathbf{E}_I$ is the submatrix $\mathbf{E}_{I,\{1,\ldots,k\}}$ of $\mathbf{E}$ and $\mathbf{X}_I = \mathbf{X}_{I,I}$, then it is not hard to show that V_I is the Pfaffian of the following $(2k+2) \times (2k+2)$ skew-symmetric matrix

$$\begin{pmatrix} 0_{k\times k} & -{}^t\mathbf{E}_I \\ \mathbf{E}_I & \mathbf{X}_I \end{pmatrix}. \tag{5.12}$$

We now examine how V_I transforms under the adjoint action of G.

Lemma 5.3. *For any $u \in \mathrm{O}(n)$, we have*

$$\mathrm{Ad}(u,1;0)V_I = \sum_{M\in T_{k+2,n}} \det(u_{MI})V_M. \tag{5.13}$$

Proof. One can verify by an easy calculation that in $\mathrm{O}(n)$,

$$\mathrm{Ad}(u)X_{ij} = \sum_{1\leqslant l<r\leqslant n} \begin{vmatrix} u_{li} & u_{lj} \\ u_{ri} & u_{rj} \end{vmatrix} X_{lr}.$$

Hence by Lemma 5.1 and Eq. (5.6) we obtain

$$\begin{aligned} \mathrm{Ad}(u,1;0)V_I &= \sum_{\substack{i<j \\ \{i,j\}\subset I}} \epsilon\big(\{i,j\}, I \setminus \{i,j\}\big)\big(\mathrm{Ad}(u)X_{ij}\big)\big(\mathrm{Ad}(u,1;0)D_{I\setminus\{i,j\}}\big) \\ &= \sum_{\substack{i<j \\ \{i,j\}\subset I}} \epsilon\big(\{i,j\}, I \setminus \{i,j\}\big) \sum_{1\leqslant l<r\leqslant n} \begin{vmatrix} u_{li} & u_{lj} \\ u_{ri} & u_{rj} \end{vmatrix} X_{lr} \sum_{L\in T_{k,n}} \det(u_{L,I\setminus\{i,j\}})D_L \end{aligned}$$

$$= \sum_{1\leqslant l<r\leqslant n} \sum_{L\in T_{k,n}} \begin{vmatrix} u_{li_1} & u_{li_2} & \cdots & u_{li_{k+2}} \\ u_{ri_1} & u_{ri_2} & \cdots & u_{ri_{k+2}} \\ u_{l_1i_1} & u_{l_1i_2} & \cdots & u_{l_1i_{k+2}} \\ \cdots & \cdots & \ddots & \cdots \\ u_{l_ki_1} & u_{l_ki_2} & \cdots & u_{l_ki_{k+2}} \end{vmatrix} X_{lr} D_L$$

(In the inner sum, we have put $L = \{l_1, \ldots, l_k\}$.)

$$= \sum_{M\in T_{k+2,n}} \det(u_{M,I}) \sum_{\{l,r\}\subset M} \epsilon\big(\{l,r\}, M\setminus\{l,r\}\big) X_{lr} D_{M\setminus\{l,r\}}$$

$$= \sum_{M\in T_{k+2,n}} \det(u_{M,I}) V_M. \qquad \square$$

Lemma 5.4. *Let $v \in \mathrm{O}(k)$. Then*

$$\mathrm{Ad}(1, v; 0) V_I = \det(v) V_I = \pm V_I. \tag{5.14}$$

Proof. From (5.10), Lemma 5.2, and the fact that $\mathrm{O}(n)$ and $\mathrm{O}(k)$ commute in G, we have

$$\begin{aligned} \mathrm{Ad}(1, v; 0) V_I &= \sum_{\substack{i<j \\ \{i,j\}\subset I}} \epsilon\big(\{i,j\}, I\setminus\{i,j\}\big)\big(\mathrm{Ad}(1, v; 0) X_{ij}\big)\big(\mathrm{Ad}(1, v; 0) D_{I\setminus\{i,j\}}\big) \\ &= \sum_{\substack{i<j \\ \{i,j\}\subset I}} \epsilon\big(\{i,j\}, I\setminus\{i,j\}\big) X_{ij}\big(\det(v) D_{I\setminus\{i,j\}}\big) = \det(v) V_I. \qquad \square \end{aligned}$$

Lemma 5.5. *For $1 \leqslant r \leqslant n$, $1 \leqslant s \leqslant k$ and $I \in T_{k+2,n}$, we have*

$$\mathrm{ad}(E_{rs}) V_I = 0. \tag{5.15}$$

Proof. Using the adjoint relation equation (5.5), and the fact that $M_{n,k} \subset G$ is abelian, we obtain

$$\mathrm{ad}(E_{rs}) V_I = \sum_{\substack{i<j \\ \{i,j\}\subset I}} \epsilon\big(\{i,j\}, I\setminus\{i,j\}\big)(\delta_{ir} E_{js} - \delta_{jr} E_{is}) D_{I\setminus\{i,j\}}.$$

If $r \notin I$, the right-hand side above obviously equals 0. So assume that $r \in I$. Then the right-hand side equals

$$\begin{aligned} &\sum_{j\in I, j>r} \epsilon\big(\{r,j\}, I\setminus\{r,j\}\big) E_{js} D_{I\setminus\{r,j\}} - \sum_{i\in I, i<r} \epsilon\big(\{i,r\}, I\setminus\{i,r\}\big) E_{is} D_{I\setminus\{i,r\}} \\ &\quad = \sum_{j\in I, j>r} \epsilon\big(\{r\}, I\setminus\{r\}\big)\epsilon\big(\{j\}, I\setminus\{r,j\}\big) E_{js} D_{I\setminus\{r,j\}} \\ &\qquad - \sum_{i\in I, i<r} -\epsilon\big(\{r\}, I\setminus\{r\}\big)\epsilon\big(\{i\}, I\setminus\{i,r\}\big) E_{is} D_{I\setminus\{i,r\}} \\ &\quad = \epsilon\big(\{r\}, I\setminus\{r\}\big) \sum_{\ell\in I,\, \ell\neq r} \epsilon\big(\{\ell\}, I\setminus\{\ell,r\}\big) E_{\ell s} D_{I\setminus\{\ell,r\}}. \end{aligned}$$

Now if we put $I = \{i_1, \dots, i_{k+2}\}$ and $r = i_a$, the last expression above is just $(-1)^{a-1}$ times the $(k+1) \times (k+1)$ determinant

$$\begin{vmatrix} E_{i_1,s} & E_{i_1,1} & E_{i_1,2} & \dots & E_{i_1,k} \\ \dots & \dots & \dots & \ddots & \dots \\ E_{i_{a-1},s} & E_{i_{a-1},1} & E_{i_{a-1},2} & \dots & E_{i_{a-1},k} \\ E_{i_{a+1},s} & E_{i_{a+1},1} & E_{i_{a+1},2} & \dots & E_{i_{a+1},k} \\ \dots & \dots & \dots & \ddots & \dots \\ E_{i_{k+2},s} & E_{i_{k+2},1} & E_{i_{k+2},2} & \dots & E_{i_{k+2},k} \end{vmatrix}$$

when expanded by minors along the first column. This determinant clearly equals 0. □

Corollary 5.6. *For any $E \in M_{n,k}$ and $I \in T_{k+2,n}$, we have*

$$\mathrm{Ad}(1, 1; E)V_I = V_I. \tag{5.16}$$

For $d = 1, \dots, k$, we define the elements

$$Q_d = \sum_{\substack{I \in T_{d,n} \\ J \in T_{d,k}}} D_{IJ}^2 \in U(\mathfrak{g}) \tag{5.17}$$

and

$$Q_{k+1} = \sum_{I \in T_{k+2,n}} V_I^2 \in U(\mathfrak{g}). \tag{5.18}$$

For each l, Q_l is an order $2l$ element of $U(\mathfrak{g})$. The results above, from Lemma 5.1 to Corollary 5.6, show that the Q_l's belong to $\mathfrak{z}(\mathfrak{g})$:

Theorem 5.7. *For $l = 1, \dots, k+1$, $Q_l \in \mathfrak{z}(\mathfrak{g})$.*

Proof. Using Corollary 5.6, we see that $\mathrm{Ad}(1, 1; E)Q_l = Q_l$ for each $E \in M_{n,k}$ and each $l = 1, \dots, k+1$. Let us now show that Q_{k+1} is invariant under $\mathrm{Ad}(u, 1; 0)$ for each $u \in \mathrm{O}(n)$. The proof that Q_l is invariant, for $l = 1, \dots, k$, is similar.

Let $W = \bigwedge^{k+2} \mathbb{R}^n$, and equip W with the inner product $\langle v_1 \wedge \cdots \wedge v_{k+2}, w_1 \wedge \cdots \wedge w_{k+2} \rangle = \det(v_i \cdot w_j)$. The standard representation of $\mathrm{O}(n)$ on $\mathbb{R}^n$ extends to a unitary representation of $\mathrm{O}(n)$ on W via the operators $\bigwedge^{k+2} u : \bigwedge^{k+2} \mathbb{R}^n \to \bigwedge^{k+2} \mathbb{R}^n$, $u \in \mathrm{O}(n)$. Letting $e_1, \dots, e_n$ be the standard basis of $\mathbb{R}^n$, we see that by Lemma 5.3, the linear map $\varphi : W \to U(\mathrm{so}(n))$, $e_I := e_{i_1} \wedge \cdots \wedge e_{i_{k+2}} \mapsto V_I$ $(I = \{i_1, \dots, i_{k+2}\} \in T_{k+2,n})$, is equivariant under $\mathrm{O}(n)$. (Here we note that $\bigwedge^{k+2}(u) \cdot e_I = \sum_J \det(u_{JI}) e_J$.) The linear map $\varphi \otimes \varphi : W \otimes W \to Uu(\mathrm{so}(n))$, given on decomposable elements by $\varphi \otimes \varphi(\omega_1 \otimes \omega_2) = \varphi(\omega_1)\varphi(\omega_2)$ is also $\mathrm{O}(n)$-equivariant. Now $\varphi \otimes \varphi(e_I \otimes e_I) = V_I^2$. Since the e_I $(I \in T_{k+2,n})$ form an orthonormal basis of W, and the tensor $\sum_{I \in T_{k+2,n}} e_I \otimes e_I$ is an $\mathrm{O}(n)$-invariant element of $W \otimes W$, it follows that $U_{2k} = \sum_{I \in T_{k+2}} V_I^2$ is also $\mathrm{O}(n)$-invariant.

An analogous argument shows that $\mathrm{Ad}(1, v; 0)Q_l = Q_l$ for all $v \in \mathrm{O}(k)$ and all $l = 1, \dots, k+1$. $\square$

We can complete the Q_l in (5.17) and (5.18) to a set of generators of $U(\mathfrak{g})$ as follows. Let d be any positive integer such that $k + 2d \leqslant n$. If $I \in T_{k+2d,n}$, let V_I be the element of $U(\mathfrak{g})$ given by

$$V_I = \sum_{\substack{J \subset I \\ \#J = k}} \epsilon(J, I \setminus J) D_J X_{I \setminus J}. \tag{5.19}$$

As in (5.10) we have put $D_J = D_{J,\{1,\dots,k\}}$ for simplicity. Note that the sum above ranges over all $J \in T_{k,n}$ contained in I, and that V_I is an element of $U(\mathfrak{g})$ of order $k + d$. It can be shown, as with (5.12), that V_I is the Pfaffian of the $(2k + 2d) \times (2k + 2d)$ skew-symmetric matrix

$$\begin{pmatrix} 0_{k \times k} & -{}^t\mathbf{E}_I \\ \mathbf{E}_I & \mathbf{X}_I \end{pmatrix} \tag{5.20}$$

whose entries are vectors in $\mathfrak{g}$. Moreover, it can be proved in a manner similar to the lemmas above that the V_I behave as follows under $\mathrm{Ad}(G)$:

$$\begin{aligned} \mathrm{Ad}(u, 1; 0)V_I &= \sum_{J \in T_{k+2d,n}} u_{JI} V_J, \quad u \in \mathrm{O}(n), \\ \mathrm{Ad}\big((1, v; 0)V_I &= \det(v) V_I, \quad v \in \mathrm{O}(k), \\ \mathrm{Ad}(1, 1; E)V_I &= V_I, \quad E \in M_{n,k}. \end{aligned} \tag{5.21}$$

Hence as in Theorem 5.7, we see that the element $Q_{k+d} = \sum_{I \in T_{k+2d,n}} V_I^2$ belongs to $\mathfrak{z}(\mathfrak{g})$. We will not need the higher order Q_l to characterize the range of the matrix Radon transform R. Nonetheless, in a subsequent paper, we will prove the following result:

Theorem 5.8. *The elements Q_l, for $l = 1, \dots, [(n + k)/2]$, form an algebraically independent set of generators of $\mathfrak{z}(\mathfrak{g})$.*

Remark 5.9. When $k = 1$, then G is just the Euclidean motion group, and the theorem above is Theorem 2.1 of [3] or Theorem 5.1 of [5].

6. Differential equations and the range of *R*

In this section we will prove our main result characterizing the range of the matrix Radon transform $R : \mathcal{S}(M_{n,k}) \to \mathcal{S}(\mathrm{St}(n, m) \times_{\mathrm{O}(m)} M_{m,k})$, when $k < m < n$. The range-characterizing operators turn out to be precisely the elements V_I of $U(\mathfrak{g})$ defined in Eq. (5.10). Equivalently, the range can be characterized by the single Casimir element Q_{k+1} defined in (5.18).

This section is organized as follows. Proposition 6.1 will establish that the differential equations $d\nu(V_I)\varphi = 0$ (for $I \in T_{k+2,n}$) are necessarily satisfied by any $\varphi \in R\mathcal{S}(M_{n,k})$. Our main result, Theorem 6.2, then shows that any $\varphi \in \mathcal{S}(\Xi)$ satisfying these equations is of the form Rf, for $f \in \mathcal{S}(M_{n,k})$. We will need several other results for the proof of our main theorem, in particular a crucial smoothness result in Theorem 6.10. The proof of Theorem 6.10 in turns relies on

a version of the weak Nullstellensatz (Theorem 6.7) for irreducible polynomials in $\mathbb{R}^n$, whose proof is postponed to Section 7. Theorem 6.10 generalizes a smoothness result for functions on $\mathbb{R}^n$ arising from smooth functions on d-planes, which can be found in [2,4,21].

Recall from Section 4 that λ and $d\lambda$ are, respectively, the left regular representation and the infinitesimal left regular representation of G and $U(\mathfrak{g})$ on $C^\infty(M_{n,k})$. If we equip $M_{n,k}$ with the standard Euclidean coordinates x_{ij} $(1 \leqslant i \leqslant n, 1 \leqslant j \leqslant k)$, it is easy to see that

$$d\lambda(E_{ij}) = -\frac{\partial}{\partial x_{ij}}, \tag{6.1}$$

$$d\lambda(X_{ij}) = \sum_{l=1}^{k} \left(x_{il} \frac{\partial}{\partial x_{jl}} - x_{jl} \frac{\partial}{\partial x_{il}} \right). \tag{6.2}$$

This leads to the following result.

Proposition 6.1. *Let $I \in T_{k+2,n}$. Then $d\lambda(V_I) = 0$.*

Proof. By (6.2) we have

$$\begin{aligned} d\lambda(V_I) &= \sum_{\substack{i<j \\ \{i,j\}\subset I}} \epsilon\bigl(\{i,j\}, I \setminus \{i,j\}\bigr)\, d\lambda(X_{ij})\, d\lambda(D_{I\setminus\{i,j\}}) \\ &= \sum_{\substack{i<j \\ \{i,j\}\subset I}} \epsilon\bigl(\{i,j\}, I \setminus \{i,j\}\bigr) \left(\sum_{l=1}^{k} \left(x_{il} \frac{\partial}{\partial x_{jl}} - x_{jl} \frac{\partial}{\partial x_{il}} \right) \right) \circ d\lambda(D_{I\setminus\{i,j\}}) \\ &= \sum_{l=1}^{k} \sum_{\substack{i<j \\ \{i,j\}\subset I}} \epsilon\bigl(\{i,j\}, I \setminus \{i,j\}\bigr) \left(x_{il} \frac{\partial}{\partial x_{jl}} - x_{jl} \frac{\partial}{\partial x_{il}} \right) \circ d\lambda(D_{I\setminus\{i,j\}}). \end{aligned}$$

We will show that each of the terms in the outer sum above vanishes. First, let us rewrite the lth term in the outer sum as $\sum_{i\in I} x_{il} E_{i,l,I}$, with

$$\begin{aligned} E_{i,l,I} = {} & \epsilon\bigl(\{i\}, I \setminus \{i\}\bigr) \sum_{j\in I, j>i} \epsilon\bigl(\{j\}, I \setminus \{i,j\}\bigr) \frac{\partial}{\partial x_{jl}} \circ d\lambda(D_{I\setminus\{i,j\}}) \\ & - \epsilon\bigl(\{i\}, I \setminus \{i\}\bigr) \sum_{j\in I, j<i} -\epsilon\bigl(\{j\}, I \setminus \{i,j\}\bigr) \frac{\partial}{\partial x_{jl}} \circ d\lambda(D_{I\setminus\{i,j\}}). \end{aligned}$$

Now if $I = \{i_1, \ldots, i_{k+2}\}$ and $i = i_a$, then one can readily see from the above that $E_{i,l,I}$ equals the determinantal differential operator

$$(-1)^{a-1}\begin{vmatrix} \partial/\partial x_{i_1,l} & -\partial/\partial x_{i_1,1} & \cdots & -\partial/\partial x_{i_1,k} \\ \cdots & \cdots & \ddots & \cdots \\ \partial/\partial x_{i_{a-1},l} & -\partial/\partial x_{i_{a-1},1} & \cdots & -\partial/\partial x_{i_{a-1},k} \\ \partial/\partial x_{i_{a+1},l} & -\partial/\partial x_{i_{a+1},1} & \cdots & -\partial/\partial x_{i_{a+1},k} \\ \cdots & \cdots & \ddots & \cdots \\ \partial/\partial x_{i_{k+2},l} & -\partial/\partial x_{i_{k+2},1} & \cdots & -\partial/\partial x_{i_{k+2},k} \end{vmatrix},$$

which equals 0. $\square$

Recall that ν and $d\nu$ denote the left regular representations of G and $U(\mathfrak{g})$, respectively, on $C^\infty(\Xi)$. By Proposition 6.1, Eq. (4.5), and the subsequent remark, we have shown that the V_I, for $I \in T_{k+2,n}$, give rise to necessary differential equations satisfied by the range of R:

$$R\mathcal{S}(M_{n,k}) \subset \{\varphi \in \mathcal{S}(\Xi) \mid d\nu(V_I)\varphi = 0\} \subset \{\varphi \in \mathcal{S}(\Xi) \mid d\nu(Q_{k+1})\varphi = 0\}. \tag{6.3}$$

The main result of this paper, given below, is that the above sets are equal.

Theorem 6.2. *Let $\varphi \in \mathcal{S}(\Xi)$ satisfy the Pfaffian system $d\nu(V_I)\varphi = 0$, for all $I \in T_{k+2,n}$. Then there exists a function $f \in \mathcal{S}(M_{n,k})$ such that $\varphi = Rf$.*

When $k = 1$, this result is just the range theorem for the d-plane transform on $\mathbb{R}^n$, where $d = n - m$, which in parametric form goes all the way back to F. John's famous 1938 paper [8]. See also [2,4,21].

Before we prove Theorem 6.2, we consider a few preliminaries. Let us first note that the last two sets in (6.3) coincide. To show this, we will need to introduce some further notation. Since G and H are unimodular, the homogeneous space $\Xi = G/H$ has a locally finite G-invariant measure $d\xi$, unique up to constant multiple. Using the G-action on Ξ given by (2.4), one can see that this measure can be written in the following way. For each $F \in \mathrm{St}(n,m)$, let $(F) \in G_{n,m}$ denote its linear span (i.e., the column space of F). If $\varphi \in L^1(\Xi)$, we have

$$\int_{\Xi} \varphi(\xi)\,d\xi = \int_{G_{n,m}} \int_{M_{m,k}} \varphi[F,B]\,dB\,d(F),$$

where $d(F)$ is the normalized $\mathrm{O}(n)$-invariant measure on $G_{n,m}$.

Since $d\xi$ is G-invariant, we see that for any $I \in T_{k+2,n}$, the formal adjoint of $d\nu(V_I)$ is $(-1)^{k+1} d\nu(V_I)$:

$$\langle d\nu(V_I)\varphi, \psi\rangle = \langle \varphi, (-1)^{k+1} d\nu(V_I)\psi\rangle$$

for $\varphi, \psi \in \mathcal{S}(\Xi)$, where $\langle\cdot,\cdot\rangle$ denotes the L^2 inner product in Ξ.

We deduce from this that if $\varphi \in \mathcal{S}(\Xi)$, the (Pfaffian) system of equations $d\nu(V_I)\varphi = 0$ ($I \in T_{k+2,n}$) is equivalent to the order $2k+2$ G-invariant differential equation $d\nu(Q_{k+1})\varphi = 0$. In fact, if $d\nu(Q_{k+1})\varphi = 0$ then

$$0 = \langle d\nu(Q_{k+1})\varphi, \varphi\rangle = (-1)^{k+1} \sum_{I \in T_{k+2,n}} \langle d\nu(V_I)\varphi, d\nu(V_I)\varphi\rangle,$$

so that $d\nu(V_I)\varphi = 0$ for all I. (The converse is, of course, trivial.) This shows that the last two sets in (6.3) are equal.

Next we use the partial Fourier transform $\mathcal{F}$ on $\varXi$ to introduce the representation $\widetilde{\nu}$ of G on $\mathcal{S}(\varXi)$ by

$$\widetilde{\nu}(g)\varPhi := \mathcal{F}\big(\nu(g)\mathcal{F}^{-1}\varPhi\big), \quad \varPhi \in \mathcal{S}(\varXi), \tag{6.4}$$

or in other words,

$$\widetilde{\nu}(g)\mathcal{F}\varphi = \mathcal{F}\big(\nu(g)\varphi\big), \quad \varphi \in \mathcal{S}(\varXi). \tag{6.5}$$

Upon differentiation, we obtain the associated (infinitesimal) representation $d\widetilde{\nu}$ of $U(\mathfrak{g})$ on $\mathcal{S}(\varXi)$, which is given by

$$d\widetilde{\nu}(D)\varPhi = \mathcal{F}\big(d\nu(D)\mathcal{F}^{-1}\varPhi\big). \tag{6.6}$$

Note that by (2.4) and (3.5), we have

$$\widetilde{\nu}(u, v; 0)\varPhi[F, C] = \nu(u, v; 1)\varPhi[F, C] \tag{6.7}$$

for all $u \in \mathrm{O}(n)$, $v \in \mathrm{O}(k)$, and

$$\widetilde{\nu}(1, 1; E)\varPhi[F, C] = e^{-i\,\mathrm{tr}({}^tEFC)}\varPhi[F, C] \tag{6.8}$$

for all $E \in M_{n,k}$. Equation (6.7) gives us

$$d\widetilde{\nu}(X_{ij})\varPhi[F, C] = d\nu(X_{ij})\,\varPhi[F, C], \quad 1 \leqslant i < j \leqslant n, \tag{6.9}$$

and from (6.8) we get

$$d\widetilde{\nu}(E)\varPhi[F, C] = -i\,\mathrm{tr}\big({}^tEFC\big)\varPhi[F, C] \tag{6.10}$$

for all $E \in M_{n,k}$. In particular, this last equation implies that

$$d\widetilde{\nu}(E_{jk})\varPhi[F, C] = -i(FC)_{jk}\varPhi[F, C]$$

so that for any $I \in T_{d,n}$ and $J \in T_{d,k}$,

$$d\widetilde{\nu}(D_{IJ})\varPhi[F, C] = (-i)^d \det\big((FC)_{IJ}\big)\varPhi[F, C]. \tag{6.11}$$

Here $(FC)_{IJ}$ denotes the $d \times d$ submatrix $((FC)_{ij})_{i\in I, j\in J}$ of FC.

If $I \in T_{k+2,n}$, we see that (6.9) and (6.11) imply that

$$\begin{aligned} &d\widetilde{\nu}(V_I)\varPhi[F, C] \\ &\quad= (-i)^k \sum_{\substack{i<j \\ \{i,j\}\subset I}} \epsilon\big(\{i, j\}, I \setminus \{i, j\}\big)\det\big((FC)_{I\setminus\{i,j\}}\big)\,d\nu(X_{ij})\,\varPhi[F, C]. \end{aligned} \tag{6.12}$$

In the above we have written (as in Section 5) $(FC)_{I\setminus\{i,j\}}$ to denote the $k \times k$ submatrix of FC whose rows are in $I \setminus \{i, j\}$.

In addition, by (6.7) or the fact that the partial Fourier transform $\mathcal{F}$ commutes with the action of $\mathrm{O}(n) \times \mathrm{O}(k)$ on Ξ, Lemmas 5.3 and 5.4 show that

$$\begin{aligned}\big(d\widetilde{\nu}(V_I)\big)^{(u,1;0)} &= \nu(u,1;0) \circ d\widetilde{\nu}(V_I) \circ \nu\big(u^{-1},1;0\big)\\ &= \sum_{M \in T_{k+2,n}} \det(u_{MI})\, d\widetilde{\nu}(V_M), \quad u \in \mathrm{O}(n) \end{aligned}\tag{6.13}$$

and

$$\begin{aligned}\big(d\widetilde{\nu}(V_I)\big)^{(1,v;0)} &= \nu(1,v;0) \circ d\widetilde{\nu}(V_I) \circ \nu\big(1,v^{-1};0\big)\\ &= \pm d\widetilde{\nu}(V_I), \quad v \in \mathrm{O}(k).\end{aligned}\tag{6.14}$$

Finally, let $\pi : \Xi \to M_{n,k}$ be the map given by $\pi[F,C] = FC$. Using local cross-sections from Ξ into $\mathrm{St}(n,m) \times M_{m,k}$, we see that π is C^∞. Since $k < m$, it is clearly surjective. π also commutes with the action of $\mathrm{O}(n) \times \mathrm{O}(k)$ on Ξ and on $M_{n,k}$:

$$\pi\big((u,v;0) \cdot \xi[F,C]\big) = (u,v;0) \cdot \pi[F,C] = uFCv^{-1}, \tag{6.15}$$

for all $u \in \mathrm{O}(n)$, $v \in \mathrm{O}(k)$, and $[F,C] \in \Xi$.

For the rest of this section, we will be proving our main result, Theorem 6.2, via a series of lemmas. Suppose then that $\varphi \in \mathcal{S}(\Xi)$ satisfies the Pfaffian system $d\nu(V_I)\varphi = 0$ for all $I \in T_{k+2,n}$. We want (eventually) to prove that $\varphi = Rf$ for some $f \in \mathcal{S}(M_{n,k})$. Let $\Phi = \mathcal{F}\varphi$. Then Φ satisfies the system $d\widetilde{\nu}(V_I)\Phi = 0$. Let us now use the expression (6.12) for $d\widetilde{\nu}(V_I)$, which shows that it is, happily, a first order differential operator on Ξ, in order to prove the following crucial lemma.

Lemma 6.3. *There exists a function h on $M_{n,k}$ such that $\Phi = h \circ \pi$.*

Remark 6.4. The function h whose existence is asserted here is not a priori smooth on $M_{n,k}$. The smoothness of h is the subject of the next lemma.

We now prove Lemma 6.3. Suppose that $X \in M_{n,k}$. We want to show that Φ is constant on the preimage $\pi^{-1}(X) = \{\xi[F,C] \mid FC = X\}$. Observe first that $\mathrm{rank}(X) = \mathrm{rank}(C)$. Now X can be expressed as a product

$$X = u \left(\begin{array}{ccc} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \\ \hline & 0_{n-k,k} & \end{array}\right) v^{-1} \tag{6.16}$$

for some $u \in \mathrm{O}(n)$ and $v \in \mathrm{O}(k)$, where $\lambda_1 \geqslant \cdots \geqslant \lambda_k \geqslant 0$, and the λ_j's are uniquely determined. Let D be the middle "diagonal" matrix on the right-hand side of (6.16).

By Eqs. (6.13), (6.14), the translate $\Psi = \Phi^{(u^{-1},v^{-1};0)}$ satisfies the system $d\widetilde{\nu}(V_I)\Psi = 0$, for all $I \in T_{k+2,n}$. Replacing Φ by Ψ if necessary, we therefore see that in order to prove Lemma 6.3, we only need to check that Φ is constant on the preimage $\pi^{-1}(D)$. We will do this separately for D of varying rank.

First let us assume that $\operatorname{rank}(D) = k$; that is to say, $\lambda_k > 0$. (This is the generic case.) Let $\xi = \xi[F, C] \in \Xi$ be such that $FC = D$. Since the column space of D is a subspace of (F), we can, by right multiplying F by an appropriate $\tau \in \mathrm{O}(m)$ (and, of course, left multiplying C by τ^{-1}), assume that

$$F = \left(\begin{array}{ccc|c} 1 & & & \\ & \ddots & & 0_{k,m-k} \\ & & 1 & \\ \hline & 0_{n-k,k} & & F' \end{array}\right), \tag{6.17}$$

where $F' \in \mathrm{St}(n-k, m-k)$. Then C must be of the form

$$C = \left(\begin{array}{ccc} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \\ \hline & 0_{m-k,k} & \end{array}\right). \tag{6.18}$$

For F and C given as above, we note that F' is arbitrary; that is, F' can be replaced by $F'\tau$, for any $\tau \in \mathrm{O}(m-k)$ (to give the same ξ). ξ thus corresponds to an element of the Grassmannian $G_{n-k,m-k}$, and this shows that $\pi^{-1}(D)$ is in one-to-one correspondence with $G_{n-k,m-k}$. Furthermore, if we dentify $\mathrm{O}(n-k)$ with the subgroup of $\mathrm{O}(n)$ given by the matrices

$$\begin{pmatrix} I_k & 0 \\ 0 & \sigma \end{pmatrix}, \quad \sigma \in \mathrm{O}(n-k),$$

we see that this Grassmannian is an orbit of $\mathrm{O}(n-k)$, and is thus an embedded submanifold of Ξ.

So fix $\xi[F, C] \in \pi^{-1}(D)$, with F and C given as in (6.17) and (6.18). Using the expression (6.12) for $d\widetilde{\nu}(V_I)$, we obtain

$$\begin{aligned} 0 &= d\widetilde{\nu}(V_I)\Phi[F, C] \\ &= (-i)^k \sum_{\substack{i<j \\ \{i,j\}\subset I}} \epsilon\big(\{i, j\}, I \setminus \{i, j\}\big) \det\big((D)_{I\setminus\{i,j\}}\big)\, d\nu(X_{ij})\, \Phi[F, C]. \end{aligned} \tag{6.19}$$

The determinant $\det(D_{I\setminus\{i,j\}})$ is nonzero only when $I \setminus \{i, j\} = \{1, \ldots, k\}$, so the equation above is not trivial only if we assume that $I = \{1, \ldots, k, r, s\}$, where $k < r < s$. For such I, (6.19) becomes

$$0 = (-i)^k \lambda_1 \cdots \lambda_k\, d\nu(X_{rs})\, \Phi[F, C]$$

and hence

$$d\nu(X_{rs})\,\Phi[F,C]=0.$$

Since $\pi^{-1}(D)\approx G_{n-k,m-k}$ is a connected orbit of $\mathrm{O}(n-k)$, and the $d\nu(X_{rs})$ (for all r and s such that $k<r<s$) span the tangent spaces of this orbit, this last equation shows that Φ is constant on the orbit $\pi^{-1}(D)$. Using (6.16) and the subsequent remarks, this also shows that Φ is constant on the preimages $\pi^{-1}(X)$, when $X\in M_{n,k}$ is of maximum rank.

In order to show that Φ is constant on $\pi^{-1}(X)$ for $\operatorname{rank}(X)<k$, we proceed by downward induction on $\operatorname{rank}(X)$. We have already proved the result for $\operatorname{rank}(X)=k$. Now assume that Φ is constant on all $\pi^{-1}(Y)$, for all $Y\in M_{n,k}$ of rank $>l$, where $0\leqslant l<k$.

Then let $X\in M_{n,k}$ be of rank l. As we have already seen, it is enough to prove this result for X in "diagonal" form

$$X=\begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_l & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \\ \hline & & & 0_{n-k,k} & & \end{pmatrix}. \tag{6.20}$$

Let $\xi[F,C]\in\pi^{-1}(X)$. Since the column space of X is a subspace of (F), we can assume (via right multiplication by an element of $\mathrm{O}(m)$) that

$$F=\left(\begin{array}{ccc|c} 1 & & & \\ & \ddots & & 0_{l,m-l} \\ & & 1 & \\ \hline & 0_{n-l,l} & & F' \end{array}\right)=\left(\begin{array}{c|c} I_l & 0_{l,m-l} \\ \hline 0_{n-l,l} & F' \end{array}\right), \tag{6.21}$$

where $F'\in\mathrm{St}(n-l,m-l)$. (That is, F' is an $(m-l)$-frame in $\mathbb{R}^{n-l}=\mathbb{R}e_{l+1}+\cdots+\mathbb{R}e_n$.) This forces C to equal

$$C=\left(\begin{array}{ccc|c} \lambda_1 & & & \\ & \ddots & & 0_{l,k-l} \\ & & \lambda_l & \\ \hline & 0_{m-l,l} & & 0_{m-l,k-l} \end{array}\right). \tag{6.22}$$

Denote the columns of F' by $F'_1,\dots,F'_{m-l}$. Since $l<k<m$, we have $m-l\geqslant 2$, so the linear span of these columns in $\mathbb{R}^{m-l}$ contains a nonzero vector of the form

$$\begin{pmatrix} * \\ 0 \\ * \\ * \\ \vdots \\ * \end{pmatrix}. \tag{6.23}$$

By right-multiplying F' by an appropriate element of $\mathrm{O}(m-l)$—which yields the same matrix plane ξ—we can therefore assume that F_1' is of the form (6.23). Now we make the following claim.

Claim. *Fix any column of F', say F_r'. Let G' be any element of $\mathrm{St}(n-l, m-l)$ whose rth column coincides with F_r': $G' = [G_1', \ldots, G_{r-1}', F_r', G_{r+1}', \ldots, G_{m-l}']$. Next let $G \in \mathrm{St}(n, m)$ be given by*

$$G = \left(\begin{array}{c|c} I_l & 0_{l,m-l} \\ \hline 0_{n-l,l} & G' \end{array}\right). \tag{6.24}$$

We claim that $\Phi[F, C] = \Phi[G, C]$.

Indeed, the claim follows from the induction hypothesis as follows. For any t, let $C_t \in M_{m,k}$ be given by $C_t = C + tE_{l+r,l+1}$:

$$C_t = \begin{pmatrix} \lambda_1 & & & 0 & 0 \\ & \ddots & & \vdots & \vdots \\ & & \lambda_l & 0 & \\ & & & 0 & 0 \\ & & & \vdots & \ddots \\ & & & & 0 \\ & & & t & \vdots \\ & & & \vdots & \\ & & & 0 & 0 \end{pmatrix}.$$

Now $FC_t = GC_t$ for all t, and $\operatorname{rank}(FC_t) = l+1$ when $t \neq 0$. Thus, by the induction hypothesis, $\Phi[F, C_t] = \Phi[G, C_t]$ for $t \neq 0$. Letting $t \to 0$, we find that $\Phi[F, C] = \Phi[G, C]$, proving the claim.

Now let $G' \in \mathrm{St}(n-l, m-l)$ be given by $G' = [F_1', G_2', \ldots, G_{m-l}']$, where

$$G_2' = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = e_{l+2}.$$

Such a G' certainly exists, since F_1' is assumed to be of the form (6.23). For this G', let the m-frame G be given by (6.24). Now from the claim, we conclude that $\Phi[F, C] = \Phi[G, C]$. Our claim, of course, applies equally well to G' as it does to F', so now let $G'' \in \mathrm{St}(n-l, m-l)$ be the frame $G'' = [e_{l+1}, e_{l+2}, \ldots, e_m]$. Note that the second columns of G' and G'' coincide. Now the m-frame $F_0 = [e_1, \ldots, e_m]$ can be written in block form as

$$F_0 = \left(\begin{array}{c|c} I_l & 0_{l,m-l} \\ \hline 0_{n-l,l} & G'' \end{array}\right).$$

Hence by the claim, we obtain $\Phi[G, C] = \Phi[F_0, C]$. Together with the above, this shows that $\Phi[F, C] = \Phi[F_0, C]$. Since the $\xi[F, C]$, where F is of the form (6.21) and C is of the form (6.22), exhaust the set $\pi^{-1}(X)$, we have proved that Φ is constant on $\pi^{-1}(X)$. This finishes the induction step and completes the proof of Lemma 6.3.

Remark 6.5. The proof of Lemma 6.3 actually shows that if a continuous function Φ on Ξ is constant on the preimages $\pi^{-1}(X)$ when $\mathrm{rank}(X) = k$, then Φ is constant on $\pi^{-1}(X)$ for all X.

Remark 6.6. When $m = k$, the Casimir elements V_I do not necessarily exist, unless $k \leqslant n-2$. In such a case, it is easy to show that $d\nu(V_I) = 0$ for all I, so the Pfaffian equations $d\nu(V_I)\varphi = 0$ are trivial. But these equations are not needed in the generic (maximum rank) case, since $\pi^{-1}(X)$ consists of a single matrix plane $\xi[F, C]$. (Simply take F to be any orthonormal basis of the column space of X. Then $C = {}^tFX$; moreover, X is the point on $\xi[F, C]$ closest to the origin.) On the other hand, a (smooth) function Φ on Ξ is not necessarily constant on the preimages $\pi^{-1}(X)$ when $\mathrm{rank}(X) < k$. This can be seen, for instance, when $k = m = 1$, in which case Ξ is just the space of codimension 1 hyperplanes in $\mathbb{R}^n$. The additional conditions needed to make Φ constant on the $\pi^{-1}(X)$ are precisely the moment conditions given in [16].

Our next objective is to prove that the function h in Lemma 6.3 is C^∞ on $M_{n,k}$. In order to do this, we will need Theorem 6.7. This theorem, which is a C^∞ version of the weak Nullstellensatz for holomorphic functions on $\mathbb{C}^n$ (see [6]), is of interest in its own right, and its proof will be taken up in Section 7.

To begin, suppose that p is a C^∞ real-valued function on an open subset $\mathcal{O}$ of $\mathbb{R}^n$. We denote the zero set $\{x \in \mathcal{O} \mid p(x) = 0\}$ by $\mathcal{V}(p)$. A point x is called *regular* (for p) if $\nabla p(x) \neq 0$; the set of regular points in $\mathcal{V}(p)$ will be denoted by $\mathcal{R}(p)$. Following the terminology of R. Thom (see Bochnak's paper [1]), we say that *p has the property of zeros in $\mathcal{O}$* if, whenever g is a real-valued C^∞ function on $\mathcal{O}$ which vanishes on $\mathcal{V}(p)$, then p divides g; that is, $g = ph$ for some C^∞ function h on $\mathcal{O}$. (Of course, this implies the same result for g complex-valued, as we can apply the property to the real and imaginary parts of g.)

Theorem 6.7. *Let $p(x)$ be a polynomial with real coefficients in $\mathbb{R}^n$, such that p is irreducible over $\mathbb{C}$. Fix an open subset $\mathcal{O}$ of $\mathbb{R}^n$. Suppose that $\mathcal{R}(p)$ is dense in $\mathcal{V}(p)$. Then p has the property of zeros in $\mathcal{O}$.*

It turns out that the converse is also true for general p. More precisely, if p is any nonzero C^∞ function having the property of zeros in $\mathcal{O}$, then $\mathcal{R}(p)$ is dense in $\mathcal{V}(p)$ [1, Proposition 1].

For us, the main purpose of the theorem is to be able to use a variant of the following corollary.

Corollary 6.8. *Suppose that g is a real-valued C^∞ function on an open subset $\mathcal{O}$ of* $\mathrm{gl}(n,\mathbb{R}) = M_{n,n}$ *such that $g(X) = 0$ whenever* $\det(X) = 0$. *Then $g(X)/\det(X)$ is C^∞ on $\mathcal{O}$.*

Proof. It is clear that, for $p(X) = \det(X)$, the hypotheses of the theorem are satisfied, since $\det(X)$ is irreducible over $\mathbb{C}$, and since the regular set $\mathcal{R}(p)$ consists of all $n \times n$ matrices of rank $n-1$. $\square$

To prove the next theorem we will also need to prove the following nearly self-evident lemma.

Lemma 6.9. *Let S be an embedded submanifold, of codimension > 0, of an open subset $\mathcal{O}$ of $\mathbb{R}^n$, and let g be a continuous function on $\mathcal{O}$ such that $g \in C^1(\mathcal{O} \setminus S)$. Suppose that the partial derivatives $\partial g/\partial x_i$ (which exist and are continuous on $\mathcal{O} \setminus S$) extend to continuous functions on $\mathcal{O}$. Then $g \in C^1(\mathcal{O})$.*

Proof. Assume that $\dim S = l < n$. We need to prove that for each $s \in S$ and each i, the partial derivative $\partial g/\partial x_i$ exists at s, and that $\partial g/\partial x_i$ is continuous on all of $\mathcal{O}$.

Fix $s_0 \in S$. There is a smooth chart $(U, y_1, \ldots, y_n)$ centered at s_0, with $U \subset \mathcal{O}$, such that with respect to the y_j coordinates, the submanifold $S \cap U$ of U is given by an l-plane not containing any of the y_j coordinate axes.

By the hypothesis on g, if V is any continuous vector field on U, then Vg (which is continuous on $U \setminus S$) extends to a continuous function on U. For each j, put $V = \partial/\partial y_j$. Then by the mean value theorem and the hypothesis on S, the partial derivative $\partial g/\partial y_j$ exists at each $s \in S \cap U$, and in fact $\partial\psi(s)/\partial y_j$ is the value at s of the continuous extension of $\partial g/\partial y_j$ to U. Hence g is C^1 on U with respect to the coordinates $y_1, \ldots, y_n$, and thus also with respect to the usual coordinates $x_1, \ldots, x_n$ of $\mathbb{R}^n$. $\square$

We are now ready to prove that the function h in Lemma 6.3 is smooth.

Theorem 6.10. *Let h be any function on $M_{n,k}$ such that $h \circ \pi$ is a C^∞ function on* $\mathrm{St}(n,m) \times_{\mathrm{O}(m)} M_{m,k}$. *Then $h \in C^\infty(M_{n,k})$.*

Remark 6.11. When $k = 1$ and $m > 1$, proofs of this theorem can be found in [2,4,21]. In this case the proof is easier, since it turns out the only real problem is to show that h is C^∞ near the origin $0 \in \mathbb{R}^n$.

Proof. The proof will be divided into several parts, and is an adaptation of the one in [4] for the case $k = 1$. Unfortunately, it is long and some parts of it are rather technical.

First let $I \in T_{k,n}$. As in Section 5, for any $X \in M_{n,k}$, we will denote by X_I the $k \times k$ submatrix $X_{I,\{1,\ldots,k\}}$ of X. Put

$$M_{n,k}^I = \{X \in M_{n,k} \mid \det(X_I) \neq 0\}.$$

Then $M_{n,k}^I$ is an open dense subset of $M_{n,k}$. The set $M_{n,k}' = \bigcup_{I \in T_{k,n}} M_{n,k}^I$ consists of all $n \times k$ real matrices of rank k. Let $\Xi' = \pi^{-1}(M_{n,k}')$. Clearly $\Xi' = \{\xi[F,C] \in \Xi \mid \mathrm{rank}(C) = k\}$, and therefore Ξ' is a dense open subset of Ξ.

We also put

$$\Xi^I = \pi^{-1}\big(M_{n,k}^I\big) \tag{6.25}$$

and

$$A^I = \Xi \setminus \Xi^I. \tag{6.26}$$

(A). *h is continuous on $M_{n,k}$ and C^∞ on $M'_{n,k}$.*

To prove that h is continuous, suppose that $\{X_r\}$ is a sequence in $M_{n,k}$ converging to a matrix X. We shall show that $h(X_r) \to h(X)$. It is enough to show that if $\{X_s\}$ is any subsequence of $\{X_r\}$, then $\{X_s\}$ itself has a subsequence $\{X_{s_j}\}$ such that $h(X_{s_j}) \to h(X)$.

For each s, choose an $F_s \in \mathrm{St}(n,m)$ and $C_s \in M_{m,k}$ such that $F_s C_s = X_s$. Since $\mathrm{St}(n,m)$ is compact, there is a subsequence $\{F_{s_j}\}$ of $\{F_s\}$ which converges to a frame $F \in \mathrm{St}(n,m)$. Then, since $C_s = {}^tF_s X_s$, we see that $C := \lim_{j\to\infty} C_{s_j} = \lim_{j\to\infty} {}^tF_{s_j} X_{s_j}$ exists. Moreover, $FC = \lim_{j\to\infty} F_{s_j} C_{s_j} = X$. But then, since $h \circ \pi$ is continuous, we have $h(X) = h\circ\pi[F,C] = \lim_{j\to\infty}(h\circ\pi)[F_{s_j}, C_{s_j}] = \lim_{j\to\infty} h(X_{s_j})$.

We now prove that h is C^∞ on $M'_{n,k}$. Let $X_0 \in M'_{n,k}$. It is enough to produce a neighborhood U of X_0 in $M'_{n,k}$ and a C^∞ section $\sigma : X \mapsto \xi[F(X), C(X)]$ of U into Ξ such that $F(X)C(X) = X$, for all $X \in U$.

For each $X \in M'_{n,k}$, let $F_X \in \mathrm{St}(n,k)$ be the orthonormal k-frame obtained from X by applying the Gram–Schmidt process to each of its columns. Then the map $X \mapsto F_X$ is C^∞. In particular, the map $H : X \mapsto (F_X)$ is a C^∞ map of $M'_{n,k}$ into $G_{n,k}$. (This latter statement is independently obvious since (F_X) is just the column space of X.)

Now there exists a neighborhood W of (F_{X_0}) in $G_{n,k}$ and a C^∞ cross-section $\eta : W \to \mathrm{St}(n, n-k)$ such that for each $\tau \in W$, $\eta(\tau)$ is an orthonormal $(n-k)$-frame in $\mathbb{R}^n$ spanning $\tau^\perp$. Put $\eta(\tau) = [\eta(\tau)_1, \ldots, \eta(\tau)_{n-k}]$. Then let $\eta'(\tau)$ denote the first $m-k$ vectors in $\eta(\tau)$.

Let $U = H^{-1}(W)$. If $X \in U$, we denote by $F(X)$ the m-frame

$$F(X) = \big[F_X, \eta'\big((F_X)\big)\big].$$

The map $X \mapsto F(X)$ is then a C^∞ map from the neighborhood U of X_0 into $\mathrm{St}(n,m)$. By construction, for each $X \in U$, the first k columns of $F(X)$ span the column space of X, so there is a unique matrix $C(X) \in M_{m,k}$ such that $X = F(X)C(X)$. ($C(X)$ is of the form

$$C(X) = \left(\frac{C'(X)}{0_{m-k,k}}\right),$$

where $C'(X)$ is a $k \times k$ matrix, but this does not matter.) We have $C(X) = {}^tF(X)X$, so the map $X \mapsto C(X)$ is a C^∞ map of U into $M_{m,k}$. We can now define $\sigma : U \to \Xi$ by

$$\sigma(X) = \xi\big[F(X), C(X)\big].$$

Clearly, σ is C^∞ and satisfies $\pi \circ \sigma(X) = X$. This finishes the proof of both claims in (A).

By the above, for each $X = (x_{ij}) \in M'_{n,k}$ the $n \times k$ matrix $\nabla h(X) = (\partial h(X)/\partial x_{ij})$ is well defined, and the map $X \mapsto \nabla h(X)$ is a C^∞ map of $M'_{n,k}$ into $M_{n,k}$. Our goal is to prove that for each $V \in M_{n,k}$, there is a C^∞ function Φ_V on Ξ such that

$$\operatorname{tr}\big({}^t\big(\nabla h(FC)\big)V\big) = \Phi_V[F, C] \tag{6.27}$$

for all $\xi[F, C] \in \Xi'$. The following argument shows that this will be enough to prove Theorem 6.10. Now any directional derivative of h on $M'_{n,k}$ is of the form $d\lambda(-V)h(X) = \operatorname{tr}({}^t(\nabla h(X))V)$. If (6.27) holds, then Φ_V is constant on the preimages $\pi^{-1}(X)$ for all $X \in M'_{n,k}$. Hence by Remark 6.5, Φ_V is constant on all preimages $\pi^{-1}(X)$, with $X \in M_{n,k}$. Thus there is a function h_V on $M_{n,k}$ such that $h_V \circ \pi = \Phi_V$. By (A), h_V is continuous on $M_{n,k}$ and C^∞ on $M'_{n,k}$. But in fact, by (6.27) $h_V(X)$ coincides with $\operatorname{tr}({}^t(\nabla h(X))V)$ for all $X \in M'_{n,k}$. This shows that any directional derivative of h, defined and smooth on $M'_{n,k}$, can be extended to a continuous function on $M_{n,k}$.

We still need to prove that, for any V, the directional derivative $d\lambda(-V)h$ actually exists at each point $X \in M_{n,k}$ of rank $< k$, and that this derivative coincides with the value of h_V at X. In other words, we need to prove that $h \in C^1(M_{n,k})$. We will do this by downward induction on ranks, using Lemma 6.9. For each $l = 0, \ldots, k$, let S_l denote the set of matrices X in $M_{n,k}$ of fixed rank l, and let $\mathcal{O}_l$ denote the set of matrices in $M_{n,k}$ of rank $\geqslant l$. Then $\mathcal{O}_l$ is an open (and dense) subset of $M_{n,k}$ and S_l is an embedded submanifold of $\mathcal{O}_l$.

We already know that h is C^∞ (hence C^1) on $\mathcal{O}_k = M'_{n,k}$. Suppose that $l < k$, and assume that h is C^1 on $\mathcal{O}_{l+1}$. We want to show that h is C^1 on $\mathcal{O}_l$. For each $V \in M_{n,k}$, the directional derivative $d\lambda(-V)h$, which by the induction hypothesis is defined and continuous on $\mathcal{O}_{l+1}$, extends to a continuous function on $\mathcal{O}_l = S_l \cup \mathcal{O}_{l+1}$. (In fact, it extends to a continuous function h_V on all of $M_{n,k}$.) From Lemma 6.9, it follows that h is C^1 on $\mathcal{O}_l$; i.e., all the partial derivatives of h exist and are continuous on $\mathcal{O}_l$. Applying this argument to $l = k-1, k-2, \ldots, 0$, we see that $h \in C^1(\mathcal{O}_0) = C^1(M_{n,k})$. Since the partials of h now exist and are continuous at all points, we have also shown, in particular, that $d\lambda(-V)h(X) = h_V(X)$ for each $X, V \in M_{n,k}$.

For each $V \in M_{n,k}$, we next replace Φ by Φ_V and h by h_V, Since $h_V \circ \pi = \Phi_V$, we can apply the same argument as above to show that $h_V \in C^1(M_{n,k})$ for all V. Hence $h \in C^2(M_{n,k})$. Repeating this argument with higher order partials then shows that $h \in C^\infty(M_{n,k})$.

We now proceed with our goal of proving Eq. (6.27). As a preliminary matter, we will first need to prove claims (B) and (C), which we will state below.

Let $p : \Xi \to G_{n,m}$, $p[F, C] = (F) \cong \xi[F, 0]$ denote the projection of the vector bundle $\Xi = \mathrm{St}(n, m) \times_{\mathrm{O}(m)} M_{m,k}$ onto its base $G_{n,m}$. The last relation corresponds to the natural identification of $G_{n,m}$ with the zero section of the vector bundle Ξ, and shows that $p[F, C]$ is just the parallel translate $\xi[F, 0]$ of $\xi[F, C]$ through the origin $0 \in M_{n,k}$.

Next, suppose that $\Lambda \in T_{m,n}$. If $X \in M_{n,k}$, we denote by X_Λ the $m \times k$ submatrix $X_{\Lambda,\{1,\ldots,k\}}$ of X. Likewise, if $F \in \mathrm{St}(n, m)$, we denote by F_Λ the $m \times m$ submatrix $F_{\Lambda,\{1,\ldots,m\}}$ of F. Note that for any $v \in \mathrm{O}(m)$, we have $(Fv)_\Lambda = F_\Lambda v$. Hence the value of $|\det(F_\Lambda)|$ depends only on $(F) \in G_{n,m}$. Let $\mathrm{St}(n, m)_\Lambda$ denote the set $\{F \in \mathrm{St}(n, m) \mid \det(F_\Lambda) \neq 0\}$ and then let $G_{n,m,\Lambda}$ denote the set $\{(F) \in G_{n,m} \mid \det(F_\Lambda) \neq 0\}$. Since the function $F \mapsto \det(F_\Lambda)$ is smooth on the manifold $\mathrm{St}(n, m)$, the set $\mathrm{St}(n, m)_\Lambda$ and its projection $G_{n,m,\Lambda}$ are open subsets of $\mathrm{St}(n, m)$ and $G_{n,m}$, respectively. It is clear that the sets $\mathrm{St}(n, m)_\Lambda$, for all $\Lambda \in T_{m,n}$, form a finite open cover of $\mathrm{St}(n, m)$. (Then, of course, the $G_{n,m,\Lambda}$ will likewise form a finite open cover of $G_{n,m}$.) Let $\Xi_\Lambda = p^{-1}(G_{n,m,\Lambda})$. Then $\Xi_\Lambda = \{\xi[F, C] \mid \det(F_\Lambda) \neq 0\}$.

We now make the following claim.

(B). *The vector bundle Ξ (over $G_{n,m}$) is a finite union of local trivial bundles $\Xi_{j,\Lambda}$ (for $j = 1, \dots, \ell$, $\Lambda \in T_{m,n}$) with trivializing map*

$$\tau_{j,\Lambda} : \xi[F, C] \mapsto \big((F), (FC)_\Lambda\big), \quad \xi[F, C] \in \Xi_{j,\Lambda}. \tag{6.28}$$

(Here $((F), (FC)_\Lambda) \in G_{n,m} \times M_{m,k}$.)

Remark 6.12. When $k = 1$, so that Ξ is the space of affine $(n - m)$-planes in $\mathbb{R}^n$, this claim corresponds to the following familiar result. For any $(n - m)$-plane ξ in $\mathbb{R}^n$, let $p(\xi)$ be the parallel plane through the origin $0 \in \mathbb{R}^n$ and let $x(\xi) = (x_1(\xi), \dots, x_n(\xi))$ be the point on ξ closest to the origin. Then Ξ is a finite union of local trivial bundles W_α over $G_{n,m}$ with the trivializing map

$$\omega_\alpha : \xi \mapsto \big(p(\xi); x_{i_1}(\xi), \dots, x_{i_m}(\xi)\big)$$

for some choice of m indices $i_1, \dots, i_m$ in $\{1, \dots, n\}$.

Let us now prove claim (B). First observe that $G_{n,m}$ can be covered by open sets W_j ($j = 1, \dots, \ell$), each of which is the base of a local trivialization Ξ_j of Ξ and each of which admits a local cross-section into $\mathrm{St}(n, m)$. More precisely, there are finitely many open sets W_j covering $G_{n,m}$ such that for each j there is a C^∞ cross-section $\sigma_j : W_j \to \mathrm{St}(n, m)$ and for which the map

$$\begin{aligned} \eta_j : W_j \times M_{m,k} &\to \Xi_j \\ (\xi, C) &\mapsto \xi\big[\sigma_j(\xi), C\big] \end{aligned} \tag{6.29}$$

is the (inverse of) a local trivializing map on $\Xi_j = p^{-1}(W_j)$. (Such an open cover is always possible on the base of an associated bundle.)

For each j and each $\Lambda \in T_{m,n}$, let us put $W_{j,\Lambda} = W_j \cap G_{n,m,\Lambda}$ and let $\Xi_{j,\Lambda} = p^{-1}(W_{j,\Lambda}) = \Xi_j \cap \Xi_\Lambda$. Then $G_{n,m}$ is covered by the open sets $W_{j,\Lambda}$, some of which could, of course, be empty. The map

$$\begin{aligned} \omega_{j,\Lambda} : W_{j,\Lambda} \times M_{m,k} &\to W_{j,\Lambda} \times M_{m,k} \\ (\xi, C) &\mapsto \big(\xi, \big(\sigma_j(\xi)\big)_\Lambda C\big) \end{aligned} \tag{6.30}$$

is a bundle isomorphism of trivial vector bundles: this is because $(\sigma_j(\xi))_\Lambda$ is an invertible $m \times m$ submatrix of the m-frame $\sigma_j(\xi)$ for each $\xi \in W_{j,\Lambda}$.

The map $\tau_{j,\Lambda}$ in (6.28) will then be just the composition, on $\Xi_{j,\Lambda}$, of the two trivializations given above:

$$\tau_{j,\Lambda} = \omega_{j,\Lambda} \circ (\eta_j|_{\Xi_{j,\Lambda}})^{-1}.$$

This finishes the proof of claim (B).

Next we make the following claim about the Ξ^I.

(C). *For any $I \in T_{k,n}$, Ξ^I is open and dense in Ξ.*

To prove that Ξ^I is dense in Ξ, we first take any $\Lambda \in T_{m,n}$ such that $I \subset \Lambda$. The set $\{Y \in M_{n,m} \mid \det(Y_\Lambda) \neq 0\}$ is dense and open in $M_{n,m}$, and from this it is not hard to see that the set $\mathrm{St}(n,m)_\Lambda$ is dense and open in $\mathrm{St}(n,m)$. Hence $G_{n,m,\Lambda}$ is dense and open in $G_{n,m}$. Since the projection $p : \Xi \to G_{n,m}$ is an open map, this implies that $p^{-1}(G_{n,m} \setminus G_{n,m,\Lambda}) = \Xi \setminus \Xi_\Lambda$ is nowhere dense in Ξ.

On each trivialization $\Xi_{j,\Lambda}$ (with Λ fixed as above), the fiber coordinates are given by $\xi[F,C] \mapsto (FC)_\Lambda$. From this it is clear that $\Xi^I \cap \Xi_{j,\Lambda}$ is dense and open in the trivial bundle $\Xi_{j,\Lambda}$. Now $\Xi_\Lambda = \bigcup_j \Xi_{j,\Lambda}$, so we see from this that $\Xi^I \cap \Xi_\Lambda$ is dense and open in Ξ_Λ. But since Ξ_Λ is dense and open in Ξ, we conclude that $\Xi^I \cap \Xi_\Lambda$ (and hence Ξ^I) is dense in Ξ.

Let us now proceed with the rest of the proof of Theorem 6.10. Suppose that $\xi[F,C] \in \Xi'$. Let $X = \pi[F,C] = FC$. If $V \in M_{n,k}$, we can use Eq. (2.4) to relate the directional derivatives of Φ and h in the direction of V, as follows:

$$\begin{aligned} d\nu(V)\,\Phi[F,C] &= \frac{d}{dt}\Phi\big[F, C - t\,{}^tFV\big]\Big|_{t=0} = \frac{d}{dt}h\big(FC - tF\,{}^tFV\big)\Big|_{t=0} \\ &= \frac{d}{dt}h\big(X - tV + t\big(V - F\,{}^tFV\big)\big)\Big|_{t=0} \\ &= d\lambda(V)h(X) - d\lambda(V_{(F)})h(X). \end{aligned} \tag{6.31}$$

In the above we have put $V_{(F)} = V - F\,{}^tFV$. Clearly, $V_{(F)}$ depends only on $(F) \in G_{n,m}$. Moreover, the maps $(F) \mapsto V_{(F)}$ and $\xi[F,C] \mapsto V_{(F)}$ from $G_{n,m}$ and Ξ, respectively, into $M_{n,k}$, are C^∞. Note that the expression

$$V = V_{(F)} + F\,{}^tFV \tag{6.32}$$

corresponds to the orthogonal decomposition $M_{n,k} = \xi[F,0] \oplus \xi[F,0]^\perp$.

From (6.31) we obtain

$$\begin{aligned} \mathrm{tr}\big({}^t\big(\nabla h(X)\big)V\big) &= -d\nu(V)\,\Phi[F,C] - d\lambda(V_{(F)})h(X) \\ &= -d\nu(V)\,\Phi[F,C] + \mathrm{tr}\big({}^t\big(\nabla h(X)\big)V_{(F)}\big) \quad (X = FC). \end{aligned} \tag{6.33}$$

The first term on the right is obviously a smooth function on Ξ, so by (6.27), all we need to do to prove Theorem 6.10 is to show that the second term on the right above corresponds to a C^∞ function on Ξ. That is, we will need to prove that there is a C^∞ function Ψ_V on Ξ such that

$$\mathrm{tr}\big({}^t\big(\nabla h(FC)\big)V_{(F)}\big) = \Psi_V[F,C] \tag{6.34}$$

for all $\xi[F,C] \in \Xi'$.

We need to introduce some additional notation. First, let $\mathbf{X}$ be the $n \times n$ matrix (5.11) with vector entries $X_{ij} \in \mathrm{so}(n)$, for $1 \leqslant i, j \leqslant n$. Then, for each $I \in T_{k,n}$, denote by $\mathbf{X}_I$ the $k \times n$ submatrix $\mathbf{X}_{I,\{1,\dots,n\}}$ of $\mathbf{X}$. Next, let $d\nu(\mathbf{X})\,\Phi[F,C]$ denote the $n \times n$ matrix $(d\nu(X_{ij})\,\Phi[F,C])$. Finally, let $d\nu(\mathbf{X}_I)\,\Phi[F,C]$ be the $k \times n$ submatrix $(d\nu(X_{ij})\,\Phi[F,C])_{i \in I, 1 \leqslant j \leqslant n}$.

The maps $\xi[F,C] \mapsto d\nu(\mathbf{X})\,\Phi[F,C]$ and $\xi[F,C] \mapsto d\nu(\mathbf{X}_I)\,\Phi[F,C]$ are, of course, C^∞ from Ξ to $M_{n,n}$ and $M_{k,n}$, respectively.

Now suppose that $\xi[F,C] \in \Xi'$, and let $X = FC$. For any $W \in \xi[F,0]$, we will show that the following crucial matrix equation holds:

$$\bigl(d\nu(\mathbf{X})\,\Phi[F,C]\bigr)\,W = X\bigl({}^t\bigl(\nabla h(X)\bigr)\bigr)\,W. \tag{6.35}$$

To prove this, we first note that by (6.15), $d\lambda(X_{ij})h(X) = d\nu(X_{ij})\,\Phi[F,C]$ for all i, j. Thus we have the matrix equation $d\nu(\mathbf{X})\,\Phi[F,C] = d\lambda(\mathbf{X})h(X)$, where the right-hand side denotes the $n \times n$ matrix $(d\lambda(X_{ij})h(X))_{1\leqslant i,j\leqslant n}$.

Hence from Eq. (6.2), the (i, j)-entry of the left-hand matrix in (6.35) equals

$$\begin{aligned}\bigl(\bigl(d\lambda(\mathbf{X})h(X)\bigr)\,W\bigr)_{ij} &= \sum_{s=1}^{n}\sum_{l=1}^{k}\biggl(x_{il}\frac{\partial h(X)}{\partial x_{sl}} - x_{sl}\frac{\partial h(X)}{\partial x_{il}}\biggr)w_{sj}\\ &= \sum_{s=1}^{n}\sum_{l=1}^{k} x_{il}\frac{\partial h(X)}{\partial x_{sl}}w_{sj} - \sum_{l=1}^{k}\frac{\partial h(X)}{\partial x_{il}}\sum_{s=1}^{n} x_{sl}w_{sj}.\end{aligned}$$

But $\sum_{s=1}^{n} x_{sl}w_{sj} = 0$ for each j, l since ${}^tXW = {}^tC\,{}^tFW = 0$ by the hypothesis $W \in \xi[F,0]$. Thus the second expression above vanishes, and clearly the first expression is just the (i, j)-entry of the $n \times k$ matrix $X\,{}^t(\nabla h(X))\,W$, proving (6.35).

In particular, it follows that (for $X = FC \in M'_{n,k}$)

$$X\,{}^t\bigl(\nabla h(X)\bigr)\,V_{(F)} = \bigl(d\nu(\mathbf{X})\,\Phi[F,C]\bigr)\,V_{(F)}. \tag{6.36}$$

Now by (6.34), we just need to prove that the function $\xi[F,C] \mapsto {}^t(\nabla h(FC))\,V_{(F)}$, defined and smooth on Ξ', and with values in $M_{k,k}$, extends to a C^∞ function on Ξ.

Let $I \in T_{k,n}$. If we pick the rows of both sides of Eq. (6.36) belonging to I, we get the $k \times k$ matrix equation

$$X_I\,{}^t\bigl(\nabla h(X)\bigr)\,V_{(F)} = \bigl(d\nu(\mathbf{X}_I)\,\Phi[F,C]\bigr)\,V_{(F)} \tag{6.37}$$

for all $\xi[F,C] \in \Xi'$, with $X = FC$. Hence if $\xi[F,C] \in \Xi^I$ (a dense open subset of Ξ by claim (C)), we get

$${}^t\bigl(\nabla h(X)\bigr)\,V_{(F)} = (X_I)^{-1}\bigl(d\nu(\mathbf{X}_I)\,\Phi[F,C]\bigr)\,V_{(F)}. \tag{6.38}$$

And so, when $I, J \in T_{k,n}$ and $\xi[F,C] \in \Xi^I \cap \Xi^J$,

$$(X_I)^{-1}\bigl(d\nu(\mathbf{X}_I)\,\Phi[F,C]\bigr)\,V_{(F)} = (X_J)^{-1}\bigl(d\nu(\mathbf{X}_J)\,\Phi[F,C]\bigr)\,V_{(F)}. \tag{6.39}$$

Let $\mathrm{Cof}(X_I)$ denote the cofactor matrix of X_I. The above equation shows that

$$\begin{aligned}&\det(X_I)\,{}^t\bigl(\mathrm{Cof}(X_J)\bigr)\bigl(d\nu(\mathbf{X}_J)\,\Phi[F,C]\bigr)V_{(F)}\\ &\quad= \det(X_J)\,{}^t\bigl(\mathrm{Cof}(X_I)\bigr)\bigl(d\nu(\mathbf{X}_I)\,\Phi[F,C]\bigr)\,V_{(F)}\end{aligned} \tag{6.40}$$

whenever $\xi[F,C] \in \Xi^I \cap \Xi^J$. But since $\Xi^I \cap \Xi^J$ is a dense open subset of Ξ, (6.40) holds on all of Ξ.

Now according to claim (B), $\Xi = \bigcup_{j,\Lambda} \Xi_{j,\Lambda}$. Consider one of the local trivial bundles $\Xi_{j,\Lambda}$. If $\xi = \xi[F, C]$ lies in $\Xi_{j,\Lambda}$, and $X = FC$, we can use X_Λ as the local fiber coordinates of ξ. On $\Xi_{j,\Lambda}$, let us choose, and fix, an $I \in T_{k,n}$ such that $I \subset \Lambda$. Then pick any $J \in T_{k,n}$ such that $J \neq I$, $J \subset \Lambda$. (Since $k < m$, such a J clearly exists.) We now make the following claim.

(D). *On the trivial bundle $\Xi_{j,\Lambda}$, the set $A^I \cap \Xi^J \cap \Xi_{j,\Lambda}$ is dense in the variety $A^I \cap \Xi_{j,\Lambda}$.*

To prove claim (D), we can assume that $\Lambda = \{1, \dots, m\}$. The fiber coordinates X_Λ on $\Xi_{j,\Lambda}$ are, of course, given by elements of $M_{m.k}$. Let $M^I_{m,k} = \{B \in M_{m,k} \mid \det(B_I) \neq 0\}$, and let V^I denote the variety $M_{m,k} \setminus M^I_{m.k}$. The claim then follows from the obvious fact that on the Euclidean space $M_{m,k}$, the set $V^I \cap M^J_{m,k}$ is dense in V^I.

Let us now complete the proof of Theorem 6.10. Consider the restriction of Eq. (6.40) to the local trivialization $\Xi_{j,\Lambda}$. (Recall that we are assuming that $I, J \subset \Lambda, I \neq J$.) Claim (D) then shows that $\det(X_J) \neq 0$ on a dense subset of the variety $A^I \cap \Xi_{j,\Lambda} = \{\xi[F, C] \in \Xi_{j,\Lambda} \mid \det(X_I) = 0\}$, and this implies that on $\Xi_{j,\Lambda}$, we have

$$\det(X_I) = 0 \quad \Rightarrow \quad {}^t\big(\mathrm{Cof}(X_I)\big)\big(d\nu(\mathbf{X}_I)\, \Phi[F, C]\big)\, V_{(F)} = 0. \tag{6.41}$$

From this and the crucial Theorem 6.7, we conclude that the function

$$\frac{1}{\det(X_I)}\, {}^t\big(\mathrm{Cof}(X_I)\big)\big(d\nu(\mathbf{X}_I)\, \Phi[F, C]\big)\, V_{(F)} \tag{6.42}$$

is an C^∞ $M_{k,k}$-valued function on $\Xi_{j,\Lambda}$. Then, from Eq. (6.38), we see that on $\Xi_{j,\Lambda} \cap \Xi^I$ (which is open dense in $\Xi_{j,\Lambda}$), (6.42) equals

$${}^t\big(\nabla h(X)\big)\, V_{(F)} \quad (X = FC). \tag{6.43}$$

Thus the matrix-valued function $\xi[F, C] \mapsto {}^t(\nabla h(FC))\, V_{(F)}$, extends to a C^∞ function on $\Xi_{j,\Lambda}$. If $I' \in T_{k,n}$ also satisfies $I' \subset \Lambda$, then Eq. (6.40) shows that (6.42) equals

$$\frac{1}{\det(X_{I'})}\, {}^t\big(\mathrm{Cof}(X_{I'})\big)\big(d\nu(\mathbf{X}_{I'})\, \Phi[F, C]\big)\, V_{(F)} \tag{6.44}$$

on the open dense subset $\Xi_{j,\Lambda} \cap \Xi^I \cap \Xi^{I'}$ of $\Xi_{j,\Lambda}$. This shows that the smooth functions (6.42) and (6.44) coincide on $\Xi_{j,\Lambda}$. Let $\boldsymbol{\Psi}_V^{\{j,\Lambda\}}$ denote this smooth function.

The locally defined smooth functions $\boldsymbol{\Psi}_V^{\{j,\Lambda\}}$ given by (6.42) agree on the intersections $\Xi_{j,\Lambda} \cap \Xi_{j',\Lambda'}$. To see this, choose any I and J in $T_{k,n}$ such that $I \subset \Lambda$ and $J \subset \Lambda'$. Then

$$\begin{aligned} &\frac{1}{\det(X_I)}\, {}^t\big(\mathrm{Cof}(X_I)\big)\big(d\nu(\mathbf{X}_I)\, \Phi[F, C]\big)\, V_{(F)} \\ &\quad = \frac{1}{\det(X_J)}\, {}^t\big(\mathrm{Cof}(X_J)\big)\big(d\nu(\mathbf{X}_J)\, \Phi[F, C]\big)\, V_{(F)} \end{aligned} \tag{6.45}$$

on the trivial bundle $\Xi_{j,\Lambda} \cap \Xi_{j',\Lambda'}$. In fact, according to (6.39) they coincide on the open dense subset $(\Xi^I \cap \Xi^J) \cap (\Xi_{j,\Lambda} \cap \Xi_{j',\Lambda'})$ of $\Xi_{j,\Lambda} \cap \Xi_{j',\Lambda'}$.

We thus define the function $\boldsymbol{\Psi}_V : \Xi \to M_{k,k}$ by putting $\boldsymbol{\Psi}_V|_{\Xi_{j,\Lambda}} = \boldsymbol{\Psi}_V^{\{j,\Lambda\}}$; that is,

$$\boldsymbol{\Psi}_V[F,C] = (X_I)^{-1}\bigl(d\nu(\mathbf{X}_I)\,\Phi[F,C]\bigr)\,V_{(F)} \tag{6.46}$$

for $\xi[F,C] \in \Xi_{j,\Lambda}$, where I is any k-index in Λ. By (6.42) and (6.45), $\boldsymbol{\Psi}_V$ is a well-defined C^∞ function on Ξ. Now define

$$\Psi_V[F,C] = \operatorname{tr}\bigl(\boldsymbol{\Psi}_V[F,C]\bigr). \tag{6.47}$$

Since this coincides with $\operatorname{tr}({}^t(\nabla h(FC))\,V_{(F)})$ on Ξ', we now see that Eq. (6.34) holds, where Ψ_V is a C^∞ function on Ξ. This finishes the proof of Theorem 6.10. $\square$

Our next objective is to prove that the function h, whose existence is asserted in Lemma 6.3, belongs to the Schwartz space $\mathcal{S}(M_{n,k})$. From Theorem 6.10, we know that h is smooth. We will be using parts of the proof of that theorem in order to prove the following result, which is surprisingly nontrivial.

Lemma 6.13. *Let h be the function given in Lemma* 6.3. *Then $h \in \mathcal{S}(M_{n,k})$.*

Proof. If $\xi[F,C] \in \Xi$ and $X = FC$, we have $\Phi[F,C] = h(X)$ by Lemma 6.3. Since $\|X\| = \|C\|$ and $\Phi \in \mathcal{S}(\Xi)$, it is clear that for any $N \in \mathbb{Z}^+$, h satisfies the condition

$$\sup_{X \in M_{n,k}} \|X\|^N \bigl|h(X)\bigr| < +\infty. \tag{6.48}$$

Let $V \in M_{n,k}$. In order to prove the lemma, it is enough to show that the function Φ_V, given by (6.27), and explicitly by the right-hand side of (6.33), belongs to $\mathcal{S}(\Xi)$. The proof of Lemma 6.10 already showed that Φ_V is C^∞. If we can prove that $\Phi_V \in \mathcal{S}(\Xi)$, the relation $-d\lambda(V)h(FC) = \Phi_V[F,C]$ shows that the directional derivative $-d\lambda(V)h$ satisfies an estimate similar to (6.48). Then put $h_V = -d\lambda(V)h$. Since $h_V \circ \pi = \Phi_V$, we can apply the same reasoning, replacing h by h_V and Φ by Φ_V to arrive at similar estimates for the second order partials of h. We can then use induction on successive partials of h to prove the lemma in general.

Clearly $d\nu(V)\,\Phi \in \mathcal{S}(\Xi)$. Thus to prove that $\Phi_V \in \mathcal{S}(\Xi)$, it suffices to prove that the function Ψ_V, given by Eqs. (6.34) and (6.47), belongs to $\mathcal{S}(\Xi)$. From the latter equation and (6.46), the explicit expression of Ψ_V on each local trivialization $\Xi_{j,\Lambda}$ is

$$\operatorname{tr}\bigl(X_I^{-1} d\nu(\mathbf{X}_I)\,\Phi[F,C]V_{(F)}\bigr) = \frac{1}{\det(X_I)}\operatorname{tr}\bigl({}^t\mathrm{Cof}(X_I)\bigl(d\nu(\mathbf{X}_I)\,\Phi[F,C]\bigr)\,V_{(F)}\bigr), \tag{6.49}$$

for any $I \in T_{k,n}$ such that $I \subset \Lambda$. Our goal is now to essentially estimate Ψ_V on each of the trivializations $\Xi_{j,\Lambda}$. In order to use these estimates, we now present an alternative characterization of $\mathcal{S}(\Xi)$, which we give below.

First, we will express Ξ as a finite union of local trivial bundles, each of which is contained in an appropriate $\Xi_{j,\Lambda}$.

Recall that the $W_{j,\Lambda}$ (for $j = 1, \dots, \ell$ and $\Lambda \in T_{m,n}$) form a finite open cover of $G_{n,m}$. There exists a refinement $\{W'_{j,\Lambda}\}$ of the open cover $\{W_{j,\Lambda}\}$, with the same index set, such that the closure of $W'_{j,\Lambda}$ is contained in $W_{j,\Lambda}$.

For any $m \times m$ matrix A, we put $\|A\| = (\mathrm{tr}({}^tAA))^{1/2}$, the Euclidean (or Hilbert–Schmidt) norm of A. The function $F \mapsto \|F_\Lambda^{-1}\|$ is continuous on $\mathrm{St}(n,m)_\Lambda$, and in addition, the value of $\|F_\Lambda^{-1}\|$ depends only on $(F) \in G_{n,m,\Lambda}$. In fact, for any $v \in \mathrm{O}(m)$, we have $\|(Fv)_\Lambda^{-1}\| = \|v^{-1}F_\Lambda^{-1}\| = \|F_\Lambda^{-1}\|$.

Hence the functions $(F) \mapsto \|F_\Lambda^{-1}\|$ and $(F) \mapsto |\det F_\Lambda|$ are continuous functions on $G_{n,m,\Lambda}$. Since the $W'_{j,\Lambda}$ have compact closure in $G_{n,m,\Lambda}$, there is a number $M > 0$ sufficiently large such that $\|F_\Lambda^{-1}\| \leqslant M$ and $|\det F_\Lambda| \geqslant 1/M$ for all $(F) \in W'_{j,\Lambda}$ and for all j, Λ.

Let $\Xi'_{j,\Lambda} = p^{-1}(W'_{j,\Lambda}) = \{\xi[F,C] \mid (F) \in W'_{j,\Lambda}\}$. Of course, we have $\Xi = \bigcup_{j,\Lambda} \Xi'_{j,\Lambda}$. $\Xi'_{j,\Lambda}$ is, of course, a sub-bundle of the trivial bundle $\Xi_{j,\Lambda}$, and according to claim (B) in the proof of Theorem 6.10, $\Xi'_{j,\Lambda}$ is isomorphic to $W'_{j,\Lambda} \times M_{m,k}$, with trivializing map $\tau_{j,\Lambda} : \xi[F,C] \mapsto ((F), (FC)_\Lambda)$.

For any j, Λ, we now define the functions of rapid decrease on $\Xi'_{j,\Lambda}$ as follows. Let $\mathcal{S}(\Xi'_{j,\Lambda})$ be the set of all $\Psi \in C^\infty(\Xi'_{j,\Lambda})$ such that for any $N \in \mathbb{Z}^+$, for any differential operator D on $G_{n,m}$, and for any constant coefficient differential operator E on $M_{m,k}$, the following condition holds

$$\sup_{((F),X_\Lambda) \in \Xi'_{j,\Lambda}} \|X_\Lambda\|^N \bigl|E_{X_\Lambda} D_{(F)}\bigl(\Psi \circ \tau_{j,\Lambda}^{-1}\bigr)\bigl((F), X_\Lambda\bigr)\bigr| < +\infty. \tag{6.50}$$

Here we have parametrized $\Xi'_{j,\Lambda}$ by the pairs $((F), X_\Lambda) \in W'_{j,\Lambda} \times M_{m,k}$. We now claim that

$$\mathcal{S}(\Xi) = \bigl\{\Psi \in C^\infty(\Xi) \bigm| \Psi|_{\Xi'_{j,\Lambda}} \in \mathcal{S}\bigl(\Xi'_{j,\Lambda}\bigr) \text{ for all } j, \Lambda\bigr\}. \tag{6.51}$$

This is an alternative characterization of $\mathcal{S}(\Xi)$, and is in fact the one used for d-planes (i.e., when $k = 1$) in [4,21]. The proof of the equivalence (6.51), which involves a few tedious computations, may be safely skipped for now and will be postponed for later.

Assuming (6.51), we see that in order to prove Lemma 6.13, we just need to show that the restriction of Ψ_V to each $\Xi'_{j,\Lambda}$, which is given explicitly by the right-hand side of (6.49), belongs to $\mathcal{S}(\Xi'_{j,\Lambda})$. In other words, we need to prove that Ψ_V satisfies the estimate (6.50) for each trivialization $\Xi'_{j,\Lambda}$.

We already know that $\Phi \in \mathcal{S}(\Xi)$. Hence, by the definition (4.4) of $\mathcal{S}(\Xi)$, each entry of the matrix-valued function $d\nu(\mathbf{X}_I)\,\Phi$ belongs to $\mathcal{S}(\Xi)$. By the assumed equivalence (6.51), we conclude that the function on $\Xi'_{j,\Lambda}$ given by

$$\det(X_I)\Psi_V[F,C] = \mathrm{tr}\bigl({}^t\mathrm{Cof}(X_I)\bigl(d\nu(\mathbf{X}_I)\,\Phi[F,C]\bigr)\,V_{(F)}\bigr) \quad (I \subset \Lambda) \tag{6.52}$$

belongs to $\mathcal{S}(\Xi'_{j,\Lambda})$. Denote this function by $\Psi_I((F), X_\Lambda)$. (Strictly speaking, we should denote it by $\Psi_I \circ \tau_{j,\Lambda}^{-1}((F), X_\Lambda)$, but we will abuse notation here.) We need to prove that if we divide this expression by $\det(X_I)$, the resulting function Ψ_V (i.e., the right-hand side of (6.49)), which we already know is a smooth function, still belongs to $\mathcal{S}(\Xi'_{j,\Lambda})$.

So fix $N \in \mathbb{Z}^+$, let D be a differential operator on $G_{n,m}$ and E a constant coefficient differential operator on $M_{m,k}$. By (6.50), $D_{(F)}\Psi_I((F), X_\Lambda)$ belongs to $\mathcal{S}(\Xi'_{j,\Lambda})$. From this, we see that the family of functions on $M_{m,k}$ given by $\{D_{(F)}\Psi_I((F), X_\Lambda) \mid (F) \in W'_{j,\Lambda}\}$ forms a bounded set

in the Schwartz space $\mathcal{S}(M_{m,k})$. For any I and J contained in Λ, $I \neq J$, (6.52) (or (6.40) in the proof of Theorem 6.10) implies that

$$\det(X_I)\Psi_J\big((F), X_\Lambda\big) = \det(X_J)\Psi_I\big((F), X_\Lambda\big). \tag{6.53}$$

Applying the differential operator $D_{(F)}$ to both sides, we get

$$\det(X_I)D_{(F)}\Psi_J\big((F), X_\Lambda\big) = \det(X_J)D_{(F)}\Psi_I\big((F), X_\Lambda\big). \tag{6.54}$$

Here we note that the differential operator $D_{(F)}$ and the multiplication operators $\det(X_I)$ and $\det(X_J)$ commute, since they act on different arguments. Now by claim (D) in the proof of Theorem 6.10, applied to the trivial bundle $\Xi'_{j,\Lambda}$, we know that $\det(X_J) \neq 0$ on a dense subset of the variety $A^I \cap \Xi'_{j,\Lambda} = \{((F), X_\Lambda) \in \Xi'_{j,\Lambda} \mid \det(X_I) = 0\}$. Hence

$$\det(X_I) = 0 \quad \Rightarrow \quad D_{(F)}\Psi_I\big((F), X_\Lambda\big) = 0 \tag{6.55}$$

on $\Xi'_{j,\Lambda}$.

Next we invoke a result of Langenbruch [9, Theorem 1.6], which we state in the following way.

Theorem 6.14. *Suppose that $p(x_1, \ldots, x_n)$ is a polynomial in $\mathbb{R}^n$ with real coefficients, irreducible over $\mathbb{R}$, and having the property of zeros in $\mathbb{R}^n$. Let $\mathcal{S}_p(\mathbb{R}^n)$ be the (closed) subspace of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ consisting of all functions which vanish on the zero set $\mathcal{V}(p)$ of p. Let L_p denote multiplication by $1/p$. Then L_p is a continuous map of $\mathcal{S}_p$ into $\mathcal{S}$.*

In fact Langenbruch's result says that L_p extends to a continuous map of $\mathcal{S}$ into $\mathcal{S}$. Note that one consequence of Langenbruch's theorem is that if $f \in C^\infty(\mathbb{R}^n)$ such that $pf \in \mathcal{S}$, then $f \in \mathcal{S}$.

Going back to the proof of Lemma 6.13, we know that the family of functions on $M_{m.k}$ given by $\{D_{(F)}\Psi_I((F), X_\Lambda) \mid (F) \in W'_{j,\Lambda}\}$ forms a bounded set in the Schwartz space $\mathcal{S}(M_{m,k})$, and by (6.55) all the functions in this family vanish on the zero set $\{X_\Lambda \in M_{m,k} \mid \det(X_I) = 0\}$. But $\det(X_I)$ is $\mathbb{C}$-irreducible and, by Theorem 6.7, satisfies the property of zeros. Hence by Theorem 6.14, the family of functions

$$\left\{ \frac{1}{\det(X_I)} D_{(F)}\Psi_I\big((F), X_\Lambda\big) \,\middle|\, (F) \in W'_{j,\Lambda} \right\} \tag{6.56}$$

forms a bounded set in $\mathcal{S}(M_{m,k})$. But by (6.49), we have $\Psi_I = \det(X_I)\Psi_V$, so we see that the family (6.56) is the same as the family of functions

$$\big\{ D_{(F)}\Psi_V\big((F), X_\Lambda\big) \,\big|\, (F) \in W'_{j,\Lambda} \big\}.$$

Applying the usual seminorms defining $\mathcal{S}(M_{m.k})$, it follows that

$$\sup_{((F), X_\Lambda) \in \Xi'_{j,\Lambda}} \|X_\Lambda\|^N \big| E_{X_\Lambda} D_{(F)}\Psi_V\big((F), X_\Lambda\big)\big| < +\infty.$$

Since D, N, and E above are arbitrary, this shows that the restriction $\Psi_V|_{\Xi'_{j,\Lambda}}$ belongs to $\mathcal{S}(\Xi'_{j,\Lambda})$. This is true for all $\Xi'_{j,\Lambda}$, so by the equivalence (6.51), it follows that $\Psi_V \in \mathcal{S}(\Xi)$.

To finish the proof of Lemma 6.13, it remains to prove (6.51). Note first that if $\xi[F,C] \in \Xi'_{j,\Lambda}$, the norms $\|C\|$ and $\|X_\Lambda\| = \|F_\Lambda C\|$ are bounded with respect to each other:

$$\|X_\Lambda\| \leqslant \|F_\Lambda\| \|C\| \leqslant \sqrt{m}\|C\|, \tag{6.57}$$

$$\|C\| \leqslant \left\|(F_\Lambda)^{-1}\right\| \|X_\Lambda\| \leqslant M\|X_\Lambda\|. \tag{6.58}$$

As a first step towards proving the equivalence (6.51), we will show that for any C^∞ function Ψ on $\Xi_{j,\Lambda}$, the derivatives $d\nu(U)\Psi$, for $U \in U(\mathfrak{g})$, can be expressed as a finite sum

$$\big(d\nu(U)\Psi\big)\circ\tau_{j,\Lambda}^{-1}\big((F),X_\Lambda\big) = \sum_\alpha p_\alpha(X_\Lambda)(E_\alpha)_{X_\Lambda}(D_\alpha)_{(F)}\big(\Psi\circ\tau_{j,\Lambda}^{-1}\big)\big((F),X_\Lambda\big), \tag{6.59}$$

where $p_\alpha(X_\Lambda)$ is a polynomial, E_α is a constant coefficient differential operator on $M_{m,k}$, and D_α is a C^∞ differential operator on $G_{n,m,\Lambda}$.

Suppose then that $\Psi \in C^\infty(\Xi)$ such that the restriction of Ψ to each $\Xi'_{j,\Lambda}$ belongs to $\mathcal{S}(\Xi'_{j,\Lambda})$. Then the conversion equation (6.59), together with the bound (6.58), proves that $\Psi \in \mathcal{S}(\Xi)$.

Next, using (2.4), we will show that if Ψ is any C^∞ function on $\Xi_{j,\Lambda}$, the derivatives $D_{(F)}E_{X_\Lambda}(\Psi\circ\tau_{j,\Lambda}^{-1})$, for any differential operator D on $\Xi_{j,\Lambda}$, can be expressed as a finite sum in terms of the action of the universal enveloping algebra $U(\mathfrak{g})$ as follows:

$$\big(D_{(F)}E_{X_\Lambda}\big(\Psi\circ\tau_{j,\Lambda}^{-1}\big)\big)\circ\tau_{j,\Lambda}[F,C] = \sum_\beta h_\beta[F,C]\,d\nu(U_\beta)\Psi[F,C]. \tag{6.60}$$

Here $U_\beta \in U(\mathfrak{g})$, and each h_β is a C^∞ function on $\Xi_{j,\Lambda}$, whose restriction to each fiber $p^{-1}((F))$ is a polynomial of fixed degree k on the fiber.

Suppose now that $\Psi \in \mathcal{S}(\Xi)$. We apply the conversion equation (6.60) to the restriction of Ψ on $\Xi_{j,\Lambda}$. Now $W'_{j,\Lambda}$ has compact closure in $W_{j,\Lambda}$, so in view of the estimate (6.57), it follows there exists a constant R such that $|h_\beta[F,C]| \leqslant R(1+\|C\|)^k$ for all $\xi[F,C] \in \Xi'_{j,\Lambda}$. This estimate on h and another application of (6.57) then shows that the restriction $\Psi|_{\Xi'_{j,\Lambda}}$ belongs to $\mathcal{S}(\Xi'_{j,\Lambda})$.

Thus we need only verify the conversion equations (6.59) and (6.60). They can be derived by straightforward, if somewhat tedious, calculations using (2.4). First, let us derive (6.59). To do this, we just need derive it for $U \in \mathfrak{g}$. If we then express any $U \in U(\mathfrak{g})$ as an algebraic combination of elements of $\mathfrak{g}$, (6.59) will follow by iteration.

Let $U = E \in M_{n,k}$. By (2.4), the left-hand side of (6.59) becomes

$$\begin{aligned}\big(d\nu(E)\Psi\big)\circ\tau_{j,\Lambda}^{-1}\big((F),X_\Lambda\big) &= d\nu(E)\Psi\big[F,(F_\Lambda)^{-1}X_\Lambda\big]\\ &= \frac{d}{dt}\Psi\big[F,(F_\Lambda)^{-1}X_\Lambda - t\,{}^tFE\big]\Big|_{t=0}\\ &= \frac{d}{dt}\big(\Psi\circ\tau_{j,\Lambda}^{-1}\big)\big((F),X_\Lambda - tF_\Lambda{}^tFE\big)\Big|_{t=0}\\ &= -\big\{F_\Lambda{}^tFE\big\}_{X_\Lambda}\big(\Psi\circ\tau_{j,\Lambda}^{-1}\big)\big((F),X_\Lambda\big).\end{aligned} \tag{6.61}$$

Here $-\{F_\Lambda{}^tFE\}$ is viewed as a vector field acting on the second argument X_Λ. Now $-F_\Lambda{}^tFE$ is a linear combination of the form $\sum_\beta c_\beta((F))E_\beta$, where $E_\beta \in M_{m.k}$, and $c_\beta \in C^\infty(G_{n,m,\Lambda})$. (6.61) thus equals $\sum_\beta c_\beta((F))E_\beta(\Psi\circ\tau_{j,\Lambda}^{-1})((F),X_\Lambda)$, which is clearly of the form of the right-hand side of (6.59).

Next let $U = X \in \mathrm{so}(n)$. Then

$$\begin{aligned}
&\big(d\nu(X)\,\Psi\big)\circ\tau_{j,\Lambda}^{-1}\big((F),X_\Lambda\big)\\
&\quad= d\nu(X)\,\Psi\big[F,F_\Lambda^{-1}X_\Lambda\big]\\
&\quad= \frac{d}{dt}\big(\Psi\big[\exp(-tX)F,F_\Lambda^{-1}X_\Lambda\big]\big)\Big|_{t=0}\\
&\quad= \frac{d}{dt}\big(\Psi\circ\tau_{j,\Lambda}^{-1}\big)\big(\exp(-tX)\cdot(F),\big(\exp(-tX)F\big)_\Lambda F_\Lambda^{-1}X_\Lambda\big)\Big|_{t=0}\\
&\quad= \frac{d}{dt}\big(\Psi\circ\tau_{j,\Lambda}^{-1}\big)\big(\exp(-tX)\cdot(F),X_\Lambda\big)\Big|_{t=0}\\
&\qquad+ \frac{d}{dt}\big(\Psi\circ\tau_{j,\Lambda}^{-1}\big)\big((F),\big(\exp(-tX)F\big)_\Lambda F_\Lambda^{-1}X_\Lambda\big)\Big|_{t=0}.
\end{aligned}$$

The first term on the right above is obviously of the form $D_{(F)}(\Psi\circ\tau_{j,\Lambda}^{-1})((F),X_\Lambda)$ where D is a (first order) differential operator acting on the first argument. (It is the vector field induced by X on $G_{n,m}$.) The second term corresponds to a vector field acting on the second argument of $\Psi\circ\tau_{j,\Lambda}^{-1}$, with coefficients depending smoothly on (F) and linearly on X_Λ. We write it as $\{-(XF)_\Lambda F_\Lambda^{-1}X_\Lambda\}_{X_\Lambda}(\Psi\circ\tau_{j,\Lambda}^{-1})((F),X_\Lambda)$. As in (6.61) the quantity in braces is understood to be a directional derivative in X_Λ at each $((F),X_\Lambda)$. The second term is clearly seen to be also in the form of the right-hand side of (6.59).

Finally, let $Y \in \mathrm{so}(k)$. Then

$$\begin{aligned}
\big(d\nu(Y)\,\Psi\big)\circ\tau_{j,\Lambda}^{-1}\big((F),X_\Lambda\big) &= d\nu(Y)\,\Psi\big[F,F_\Lambda^{-1}X_\Lambda\big]\\
&= \frac{d}{dt}\big(\Psi\big[F,F_\Lambda^{-1}X_\Lambda\exp(tY)\big]\big)\Big|_{t=0}\\
&= \frac{d}{dt}\big(\Psi\circ\tau_{j,\Lambda}^{-1}\big)\big((F),X_\Lambda\exp(tY)\big)\Big|_{t=0}\\
&= \{X_\Lambda Y\}_{X_\Lambda}\big(\Psi\circ\tau_{j,\Lambda}^{-1}\big)\big((F),X_\Lambda\big),
\end{aligned}$$

in the notation above, where the last quantity in braces again denotes a directional derivative acting on the second argument. It is clearly a vector field in X_Λ with coefficients depending linearly in X_Λ, and is thus also of the form of the right-hand side of (6.59).

To derive Eq. (6.60), it is sufficient to consider the operators $D_{(F)}$ and E_{X_Λ} separately, and, in fact, to assume that these are C^∞ vector fields on $G_{n,m,\Lambda}$ and $M_{m,k}$. The reason is that we can apply induction on the order of the operator $D_{(F)}E_{X_\Lambda}$, keeping in mind that from (6.59), if $h_\beta[F,C]$ is a polynomial in C, then so is $d\nu(U)\,h_\beta[F,C]$, for any $U \in U(\mathfrak{g})$.

So let $\Psi \in C^\infty(\Xi_{j,\Lambda})$. Fix a vector $V \in M_{m,k}$ and put $E_{X_\Lambda} = V$. For any $\xi[F, C] \in \Xi_{j,\Lambda}$, we have

$$\frac{d}{dt}\big(\Psi \circ \tau_{j,\Lambda}^{-1}\big)\big((F), F_\Lambda C + tV\big)\Big|_{t=0} = \frac{d}{dt}\Psi\big[F, F_\Lambda^{-1}(F_\Lambda C + tV)\big]\Big|_{t=0}$$
$$= \frac{d}{dt}\Psi\big[F, C + t\,{}^tF\big(FF_\Lambda^{-1}V\big)\big]\Big|_{t=0}. \qquad (6.62)$$

Clearly $FF_\Lambda^{-1}V$ is a finite linear combination $\sum_\beta c_\beta((F))E_\beta$ of vectors in $M_{n,k}$, where the coefficients $c_\beta((F))$ are smooth functions of $(F) \in G_{n,m,\Lambda}$. Thus by (2.4), the right-hand side of (6.62) equals

$$-\sum_\beta c_\beta\big((F)\big)d\nu(E_\beta)\,\Psi[F, C].$$

This is in the form of the right-hand side of (6.60).

Next we verify (6.60) for $E = 1$ and for $D_{(F)}$ a vector field on $G_{n,m,\Lambda}$. Such vector fields are smooth linear combinations of the vector fields induced by the infinitesimal $\mathrm{O}(n)$-action on $G_{n,m}$. Thus it suffices to calculate

$$\frac{d}{dt}\big(\Psi \circ \tau_{j,\Lambda}^{-1}\big)\big(\exp(tX)\cdot(F), F_\Lambda C\big)\Big|_{t=0}$$

for $X \in \mathrm{so}(n)$ and $\xi[F, C] \in \Xi_{j,\Lambda}$. The above equals

$$\frac{d}{dt}\Psi\big[\exp(tX)\cdot F, \big(\exp(tX)\cdot F\big)_\Lambda^{-1} F_\Lambda C\big]\Big|_{t=0}$$
$$= \frac{d}{dt}\Psi\big[\exp(tX)\cdot F, C\big]\Big|_{t=0} + \frac{d}{dt}\Psi\big[F, \big(\exp(tX)\cdot F\big)_\Lambda^{-1} F_\Lambda C\big]\Big|_{t=0}. \qquad (6.63)$$

The first term on the right in (6.63) is just $-d\nu(X)\,\Phi[F, C]$. To find an expression for the second term, we can assume with loss of generality that $X = X_{ij}$. If neither i nor j is in Λ, then $(\exp(tX_{ij})\cdot F)_\Lambda = F_\Lambda$, so the second term on the right in (6.63) vanishes. For the other cases, let us rewrite the second term of (6.63) as

$$\frac{d}{dt}\Psi\big[F, C + {}^tFF\big\{\big(\exp(tX_{ij})\cdot F\big)_\Lambda^{-1} F_\Lambda C - C\big\}\big]\Big|_{t=0}. \qquad (6.64)$$

If both i and j are in Λ, then (6.64) equals

$$\frac{d}{dt}\Psi\big[F, C + {}^tFF\big\{F_\Lambda^{-1}\exp\big(-t(X_{ij})_\Lambda\big)F_\Lambda C - C\big\}\big]\Big|_{t=0}, \qquad (6.65)$$

where $(X_{ij})_\Lambda$ denotes the $m \times m$ submatrix of X_{ij} whose rows and columns belong to Λ. The curve on $M_{n,k}$ given by $t \mapsto H(t, [F, C]) := F\{F_\Lambda^{-1}\exp(-t(X_{ij})_\Lambda)F_\Lambda C - C\}$ has derivative at $t = 0$ given by $-FF_\Lambda^{-1}(X_{ij})_\Lambda F_\Lambda C$. This is a finite linear combination of the form

$-\sum_\beta h_\beta[F,C]V_\beta$, where $V_\beta \in M_{n,k}$ and the coefficients $h_\beta[F,C]$ are smooth functions on the trivialization $\Xi_{j,\Lambda}$ which depend linearly on C. (6.64) thus equals

$$\sum_\beta h_\beta[F,C]\,d\nu(V_\beta)\,\Psi[F,C],$$

which is of the form of the right-hand side of (6.60). Finally, let us consider the case when exactly one of the two indices i, j belongs to Λ. We can assume that $i \in \Lambda$, $j \notin \Lambda$; the proof for the case $i \notin \Lambda$, $j \in \Lambda$ is similar. In this case, the $m \times m$ matrix $(\exp(-tX_{ij})F)_\Lambda^{-1}$ can be written as

$$\big(\cos t \det F_\Lambda - \sin t \det(F_{(\Lambda\setminus\{i\})\cup\{j\}})\big)^{-1}\,{}^t\mathrm{Cof}\big(\exp(-tX_{ij})F\big)_\Lambda.$$

Hence, by the quotient rule, the curve on $M_{n,k}$ given by

$$t \mapsto \Gamma_\Lambda\big(t,[F,C]\big) := F\big\{\big(\exp(-tX_{ij})F\big)_\Lambda^{-1}F_\Lambda C - C\big\}$$

has derivative at $t = 0$ given by an expression of the form

$$\frac{1}{(\det F_\Lambda)^2}A_\Lambda[F,C], \tag{6.66}$$

where $A_\Lambda[F,C]$ is an $M_{n,k}$-valued smooth function on $\Xi_{j,\Lambda}$ which is in fact a polynomial expression in the matrix entries of F and is linear in C. (6.66) can be written as a finite sum

$$\frac{1}{(\det F_\Lambda)^2}\sum_\gamma g_\gamma[F,C]E_\gamma.$$

The second term of (6.63) thus equals

$$\frac{1}{(\det F_\Lambda)^2}\sum_\gamma g_\gamma[F,C]d\nu(E_\gamma)\,\Psi[F,C]. \tag{6.67}$$

Again this is of the form of the right-hand side of (6.60). (Note that the $(\det F_\Lambda)^2$ in the denominator is not a problem in the estimation of the right-hand side, since $|\det F_\Lambda| > 1/M$ on $W'_{j,\Lambda}$.) We have thus proved (6.60); this finishes the proof of Lemma 6.13. □

We are now in a position to complete the proof of the main result of this paper, Theorem 6.2. From here, the proof is more or less canonical.

Proof of Theorem 6.2. We started with a function $\varphi \in \mathcal{S}(\Xi)$ satisfying the Pfaffian system $d\nu(V_I)\varphi = 0$, for all $I \in T_{k+2,n}$. We then took the partial Fourier transform $\Phi = \mathcal{F}\varphi$, and, by Lemma 6.3, concluded that there exists a function h on $M_{n,k}$ such that $h(FC) = \Phi[F,C]$, for all $\xi[F,C] \in \Xi$. By Theorem 6.10 and Lemma 6.13, we proved that $h \in \mathcal{S}(M_{n,k})$. Let us denote the inverse Fourier transform of h in $M_{n,k}$ by f. (Thus $f \in \mathcal{S}(M_{n,k})$ and $\widetilde{f} = h$.) Then for any $\xi[F,C] \in \Xi$, we have $\mathcal{F}\varphi[F,C] = h(FC) = \widetilde{f}(FC) = \mathcal{F}(Rf)[F,C]$, with the last equality coming from the Fourier slice theorem (3.6). Since the partial Fourier transform $\mathcal{F}$ is injective (in fact, bijective) on $\mathcal{S}(\Xi)$, we now conclude that $Rf = \varphi$. This completes the proof of Theorem 6.2. □

7. A weak Nullstellensatz for polynomials on $\mathbb{R}^n$

In this section we supply the proof of Theorem 6.7. The local holomorphic version (and, of course, the algebraic version) of this theorem in $\mathbb{C}^n$ is well known, and is a consequence of the Weierstrass Division Theorem (see [6, p. 11]).

If we take the dualized Fourier transform version of Langenbruch's theorem (Theorem 6.14 and the following remarks), we obtain the following useful result on global solvability of partial differential operators, which also appears in Theorem 1.6 of Langenbruch's paper [9]. Suppose that p is a polynomial on $\mathbb{R}^n$, with real coefficients, $\mathbb{R}$-irreducible, and having the property of zeros. Then the corresponding partial differential operator $p(-i\partial/\partial x_1, \dots, -i\partial/\partial x_n)$ admits a continuous right inverse in the space $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions in $\mathbb{R}^n$.

Now to prove Theorem 6.7, we will need to introduce some additional terminology. As in Section 6, we first assume that $\mathcal{O}$ is an open subset of $\mathbb{R}^n$ and that p is a real-valued C^∞ function on $\mathcal{O}$. Recall that a point $x \in \mathcal{O}$ is a *critical point* of p if $\nabla p(x) = 0$. The set of critical points in $\mathcal{V}(p)$ is then $\mathcal{S}(p) = \mathcal{V}(p) \setminus \mathcal{R}(p)$.

If $x \in \mathcal{R}(p)$, then p acts as a coordinate in an appropriate chart about x, so that on that chart, p has the property of zeros. Thus the problem of divisibility by p arises in the vicinity of the points $x \in \mathcal{S}(p)$.

Now let $a \in \mathcal{O}$. We denote the germ of a C^∞ function f defined on a neighborhood of a by f_a. Likewise, the germ of a subset $N \subset \mathbb{R}^n$ at the point a will be denoted by N_a. Next let $\mathfrak{I}_a$ denote the ring of germs of real-valued analytic functions at a. Finally, we put

$$\Sigma_a(p) = \{\psi \in \mathfrak{I}_a \mid \psi^{-1}(0) \subset \mathcal{S}(p)_a\}. \tag{7.1}$$

Thus $\Sigma_a(p)$ essentially consists of those analytic functions defined near a whose zero sets in appropriately small neighborhoods of a lie in $\mathcal{S}(p)$.

Assume now that p is analytic and real-valued on $\mathcal{O}$. In [1, Théorème 1 and Corollaire 1], Bochnak provides a necessary and sufficient condition for p to have the property of zeros. Below, we summarize his results in the form relevant to us.

Theorem 7.1. (Bochnak [1]) *Suppose that p is a real-valued analytic function defined on $\mathcal{O}$. Then p has the property of zeros if and only if the following two conditions are satisfied*:

(1) *$\mathcal{R}(p)$ is dense in $\mathcal{V}(p)$.*
(2) *Suppose that $a \in \mathcal{S}(p)$, $\phi \in \mathfrak{I}_a$, and $\psi \in \Sigma_a(p)$. If p_a divides $\psi\phi$ in $\mathfrak{I}_a$, then p_a divides ϕ in $\mathfrak{I}_a$.*

Interestingly, this theorem says that if p is an analytic function, the question of C^∞ divisibility by p essentially reduces to a question of analytic divisibility by p at its singular points. (See [10].)

We now proceed with the proof of Theorem 6.7. Let us assume, then, that p is a polynomial with real coefficients, irreducible over $\mathbb{C}$, and that $\mathcal{R}(p)$ is dense in $\mathcal{V}(p)$. In order to show that p has the property of zeros, we just need to verify that p satisfies condition (2) in the above theorem.

Let $a \in \mathcal{S}(p)$. For convenience, we will identify germs of analytic functions at a with functions defined on appropriately small neighborhoods of a, shrinking their domains of definition whenever necessary.

Assume then that $\psi \in \Sigma_a(p)$, $\phi \in \mathfrak{I}_a$, and that p_a divides $\psi\phi$ in $\mathfrak{I}_a$. From the preceding remark, we can thus assume that ϕ and ψ are defined, and given by Taylor series expansions on a small ball $B = B_r(a)$ of radius r about the center a; that the zero set $\{x \in B_r(a) \mid \psi(x) = 0\}$ lies in $S(p)$; and that p divides $\psi\phi$ in the ring of analytic functions on B. Shrinking r if necessary, we can extend p, ψ, and ϕ to be holomorphic functions on the complex ball $B^{\mathrm{c}} = B_r^{\mathrm{c}}(a) \subset \mathbb{C}^n$. Let $\mathcal{V}^{\mathrm{c}}(p)$ be the holomorphic variety corresponding to p in B^{c}: $\mathcal{V}^{\mathrm{c}}(p) = \{z \in B^{\mathrm{c}} \mid p(z) = 0\}$, and let $\mathcal{R}^{\mathrm{c}}(p)$ be the set of regular points in $\mathcal{V}^{\mathrm{c}}(p)$: $\mathcal{R}^{\mathrm{c}}(p) = \{z \in \mathcal{V}^{\mathrm{c}}(p) \mid \nabla p(z) \neq 0\}$. Since p is irreducible, we can shrink r further so that $\mathcal{V}^{\mathrm{c}}(p)$ is an irreducible holomorphic variety [6, p. 12], and therefore $\mathcal{R}^{\mathrm{c}}(p)$ is connected [6, p. 21]. Moreover, again since p is irreducible, $\mathcal{R}^{\mathrm{c}}(p)$ is dense in $\mathcal{V}^{\mathrm{c}}(p)$.

Now from the hypothesis on ψ and ϕ, we see that ϕ vanishes on the $(n-1)$-dimensional real analytic submanifold $\mathcal{R}(p) \cap B$ of B. But $\mathcal{R}(p) \cap B$ is a real space of the complex submanifold $\mathcal{R}^{\mathrm{c}}(p)$ of B^{c}. That is to say, for each $x \in \mathcal{R}(p) \cap B$, the inclusion $\mathcal{R}(p) \cap B \subset \mathcal{R}^{\mathrm{c}}(p)$ appears near x as the inclusion $\mathbb{R}^{n-1} \subset \mathbb{C}^{n-1}$. (This can most easily be seen from the proof of the holomorphic implicit function theorem.) Since ϕ is holomorphic on B^{c} (and hence in $\mathcal{R}^{\mathrm{c}}(p)$), and vanishes on the real space $\mathcal{R}(p) \cap B$, and since $\mathcal{R}^{\mathrm{c}}(p)$ is connected, we see that ϕ vanishes identically on $\mathcal{R}^{\mathrm{c}}(p)$. By density, ϕ therefore vanishes on $\mathcal{V}^{\mathrm{c}}(p)$. But p is irreducible over $\mathbb{C}$, so by the local weak Nullstellensatz for holomorphic functions [6, Corollary, p. 11], we conclude that $\phi(z)/p(z)$ is a holomorphic function on B^{c}. It follows by restriction that $\phi(x)/p(x)$ is a (real-valued) analytic function on B. This shows that condition (2) in Theorem 7.1 holds, and therefore p has the property of zeros. This finishes the proof of Theorem 6.7.

Acknowledgments

We are indebted to several colleagues who have discussed with us various aspects of this paper, especially professors S. Helgason, E.T. Quinto, L. Tu, and J. Boman.

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