JOURNAL OF FUNCTIONAL ANALYSIS 49, 186–197 (1982)

The Weyl Bundle

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Communicated by A. Connes

Received April 29, 1982

Let F be a symplectic vector bundle over a space X. We construct a bundle of elementary C^*-algebras over X, and prove that the Dixmier–Douady invariant of this bundle is zero. The underlying Hilbert bundles, with their associated module structures, determine a characteristic class: we prove that this class is the second Stiefel–Whitney class of F.

INTRODUCTION

A symplectic vector bundle over a paracompact space X is a pair (F, ω), where (i) F is a real vector bundle over X with even fibre-dimension and (ii) ω is a continuous cross section of the exterior square $F \wedge F$ such that ω_x is non-degenerate at each point x in X. With these data, we construct a bundle of elementary C^*-algebras over X. We call this the Weyl bundle $W(F, \omega)$: the fibre at x is the $L_1 - C^*$-algebra k of compact operators. Let δ be the Dixmier–Douady invariant of $W(F, \omega)$, which is a cohomology class in $H^3(X; \mathbb{Z})$. We first prove

THEOREM 1. $\delta = 0$.

According to Theorem 1, there exists an underlying Hilbert bundle S such that $W(F, \omega)$ is the associated bundle of elementary C^*-algebras. In this case S is an irreducible $W(F, \omega)$-module. Let S^* denote the dual module. Then $L(S)$, the bundle of $W(F, \omega)$-equivariant maps from S to S^*, is a complex Hermitian line bundle. Let κ denote the mod 2 reduction of the first Chern class of $L(S)$; it is shown in [4] that κ is a characteristic class in $H^2(X; \mathbb{Z}/2)$. Let $w_2(F)$ be the second Stiefel–Whitney class of F. We prove

THEOREM 2. $\kappa = w_2(F)$.

The bundle S is called a bundle of symplectic spinors. Our construction is a variant of Kostant's construction [5]. It follows immediately from Theorem 1 that a bundle of symplectic spinors always exists.

0022-1236/82/140186–12\$02.00/0

The Weyl bundle is an exact analogue, for symplectic vector bundles, of the complex Clifford bundle for Euclidean vector bundles E of even fibre-dimension. For the Clifford bundle we have, as shown in [4],

(i) $\delta = \beta(w_2(E))$,

(ii) $\kappa = w_2(E)$ (when $\delta = 0$),

where β is the Bockstein determined by the exact coefficient sequence

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2 \to 0.$$

1. The Oscillator Representation and the Weyl Bundle

We begin by following closely the exposition of Borel and Wallach [1, p. 228]. We look upon $\mathbb{R}^{2n}$ as the space of all columns

$$\begin{bmatrix} x \\ y \end{bmatrix}, \qquad x, y \in \mathbb{R}^n,$$

and define

$$\beta\left(\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x' \\ y' \end{bmatrix}\right) = \langle x, y'\rangle - \langle y, x'\rangle,$$

where $\langle x, y\rangle = \sum x_i y_i$ for $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n)$.

The Heisenberg group of dimension $2n+1$ is the group with underlying space $\mathbb{R}^{2n} \times \mathbb{R}$ and multiplication given by

$$(z, t) \cdot (w, s) = (z + w, t + s + \tfrac{1}{2}\beta(z, w)).$$

We denote this Lie group by H_n.

The Stone–von Neumann theorem says that H_n has (up to dilation and duality) one infinite-dimensional, irreducible, unitary representation $(\pi, L^2(\mathbb{R}^n))$ with

$$\left(\pi\left(\begin{bmatrix} x \\ y \end{bmatrix}, t\right) f\right)(z) = \exp(i(t + \langle x, z - \tfrac{1}{2}y\rangle)) f(z + y)$$

for $f \in L^2(\mathbb{R}^n)$, $x, y \in \mathbb{R}^n$, $t \in \mathbb{R}$.

Let the unitary group $U(L^2(\mathbb{R}^n))$ of $L^2(\mathbb{R}^n)$ be given the strong operator topology. The map

$$\mathbb{R}^{2n} \to U(L^2(\mathbb{R}^n)), \tag{1.1}$$

given by $x \mapsto \pi(x, 0)$, is a homeomorphism of $\mathbb{R}^{2n}$ onto its image.

Let

$$G = \{ g \in U(L^2(\mathbb{R}^n)) \colon g\pi(z, t)\, g^{-1} = \pi(z', t), (z \in \mathbb{R}^{2n}, t \in \mathbb{R})\}.$$

In this definition, z' clearly depends on z and g. Now (1.1) implies that $z' = \nu(g)\, z$ with $\nu(g) \colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ a homeomorphism.

Let $\mathrm{Sp}(n, \mathbb{R})$ denote the symplectic group. That is,

$$\mathrm{Sp}(n, \mathbb{R}) = \{ g \in GL(2n, \mathbb{R}) \colon \beta(g \cdot v, g \cdot w) = \beta(v, w), \text{ for } v, w \in \mathbb{R}^{2n}\}.$$

$$\nu(g) \in \mathrm{Sp}(n, \mathbb{R}) \qquad \text{for } g \in G. \tag{1.2}$$

PROPOSITION 1.3. *Let $T = \{z \in \mathbb{C} \colon |z| = 1\}$. Then the sequence*

$$1 \to T \to G \xrightarrow{\nu} \mathrm{Sp}(n, \mathbb{R}) \to 1$$

is exact where $T \to G$ is the map $z \mapsto zI$.

1.4. G is a Lie group.

1.5. The *metaplectic group* is the commutator group, $\mathrm{Mp}(n, \mathbb{R})$, of G.

1.6. Set $j = \nu \mid \mathrm{Mp}(n, \mathbb{R})$. Then

$$j \colon \mathrm{Mp}(n, \mathbb{R}) \to \mathrm{Sp}(n, \mathbb{R})$$

is a double covering.

We look upon $(\mathrm{Mp}(n, \mathbb{R}), j)$ as an abstract covering group of $\mathrm{Sp}(n, \mathbb{R})$. The realization $\mathrm{Mp}(n, \mathbb{R}) \subset U(L^2(\mathbb{R}^n))$ will be denoted $(\mu, L^2(\mathbb{R}^n))$. It is called the *oscillator* (sometimes Shale–Weil, harmonic) representation of $\mathrm{Mp}(n, \mathbb{R})$.

Let $PU(H)$ be the projective unitary group $U(H)/U(1)$ with the quotient topology. The map $\mu \colon G \to U(H)$ induces a projective unitary representation $\mathrm{Sp}(n, \mathbb{R}) \to PU(H)$. This determines a commutative diagram of topological groups, with exact rows:

$$\begin{tikzcd} 1 \arrow[r] & T \arrow[r] \arrow[d, equal] & G \arrow[r] \arrow[d] & \mathrm{Sp}(n, \mathbb{R}) \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & T \arrow[r] & U(H) \arrow[r] & PU(H) \arrow[r] & 1 \end{tikzcd} \tag{1.7}$$

Notation. $T = \{z \in \mathbb{C} \colon |z| = 1\}$, the circle group.

Set $\pi(x) = \pi(x, 0)$. Define

$$\pi(f) = \int f(x)\, \pi(x)\, dx, \qquad f \in L^1(\mathbb{R}^{2n}),$$

where dx is Lebesgue measure on $\mathbb{R}^{2n}$. Then

$$\begin{aligned} g\pi(f)\,g^{-1} &= g\cdot\int f(x)\,\pi(x)\,dx\cdot g^{-1} \\ &= \int f(x)\,g\pi(x)\,g^{-1}\,dx \\ &= \int f(x)\,\pi(\nu(g)\,x)\,dx \\ &= \int f(\nu(g)^{-1}\,y)\,\pi(y)\,dy, \qquad y=\nu(g)\,x, \end{aligned}$$

since $\nu(g)\in \mathrm{Sp}(n,\mathbb{R})$ and $\det\sigma = 1$ for all $\sigma\in\mathrm{Sp}(n,\mathbb{R})$. If we define

$$(\sigma\cdot f)(y)=f(\sigma^{-1}y), \qquad f\in L^1, y\in\mathbb{R}^{2n}, \sigma\in\mathrm{Sp}(n,\mathbb{R}),$$

then we have shown that

$$g\pi(f)\,g^{-1}=\pi(\nu(g)f), \qquad g\in G. \tag{1.8}$$

Following Kastler [6, p. 17], we define the $L_1 - C^*$-algebra of $(\mathbb{R}^{2n},\beta)$ to be the C^*-algebra generated by the operators

$$\{\pi(f): f\in L^1(\mathbb{R}^{2n})\}.$$

The $L_1 - C^*$-algebra of $(\mathbb{R}^{2n},\beta)$ is denoted $\overline{L_1(\mathbb{R}^{2n})}$. Then $\mathrm{Sp}(n,\mathbb{R})$ acts as automorphisms of $L_1(\mathbb{R}^{2n})$.

Notation. We shall write

(i) $H=L^2(\mathbb{R}^n)$,

(ii) $k = C^*$-algebra of compact operators on $L^2(\mathbb{R}^n)$.

Now the $L_1 - C^*$-algebra $\overline{L_1(\mathbb{R}^{2n})}$ is a separable, infinite-dimensional C^*-algebra of operators on $L^2(\mathbb{R}^n)$ which has a unique irreducible representation (up to unitary equivalence): consequently, as in Kastler [6, Theorem 19], we have

$$\overline{L_1(\mathbb{R}^{2n})}=k.$$

It follows that the automorphism group of $\overline{L_1(\mathbb{R}^{2n})}$ is isomorphic to the projective unitary group of H:

$$PU(H)\cong \mathrm{Aut}\,\overline{L_1(\mathbb{R}^{2n})}.$$

This isomorphism is induced by the map $v \mapsto \operatorname{Ad} v$, with v in the unitary group $U(H)$.

DEFINITION. Let (F, ω) be a symplectic vector bundle over X, and let P be the principal bundle of symplectic frames. The Weyl bundle is the associated bundle

$$W(F, \omega) = P \underset{\mathrm{Sp}(n, \mathbb{R})}{\times} \overline{L_1(\mathbb{R}^{2n})}.$$

The Weyl bundle is a bundle of elementary C^*-algebras, i.e., each fibre is isomorphic to the C^*-algebra k. More precisely, a choice of symplectic frame in (F_x, ω_x) determines a definite isomorphism

$$(F_x, \omega_x) \cong (\mathbb{R}^{2n}, \beta)$$

and a definite isomorphism

$$W_x(F, \omega) \cong k.$$

Here, $W_x(F, \omega)$ is the fibre at x of the Weyl bundle.

The operator

$$\pi(f) = \int f(x)\, \pi(x)\, dx$$

is exactly the Fourier–Weyl transform of the L^1-function f. This is because

$$\pi(x) = \exp i\{x_1 Q_1 + \cdots + x_n Q_n + y_1 P_1 + \cdots + y_n P_n\},$$

where Q_j and P_j are the standard operators in quantum mechanics. Thus the C^*-algebra $\overline{L_1(\mathbb{R}^{2n})}$ is generated by the Fourier–Weyl transforms of functions in $L_1(\mathbb{R}^{2n})$. The Fourier–Weyl transform made its appearance in Weyl's classic article [8, p. 27]. This would seem to justify our terminology "Weyl bundle."

Now bundles of elementary C^*-algebras over X are classified, up to isomorphism, by the integral cohomology classes in $H^3(X; \mathbb{Z})$. These classes are the Dixmier–Douady invariants. The Dixmier–Douady invariant of the bundle A is denoted $\delta(A)$. We shall prove, by considering the oscillator representation, that the Dixmier–Douady invariant of the Weyl bundle is zero. The Dixmier-Douady invariant made its appearance in [2].

2. Proof of Theorem 1

The commutative diagram (1.7) of topological groups determines, according to Frenkel [3, p. 162], a commutative diagram of pointed sets in Čech cohomology:

$$
\begin{tikzcd}
H^1(X; T) \arrow[r] \arrow[d, equal] & H^1(X; G) \arrow[r] \arrow[d] & H^1(X; \mathrm{Sp}(n, \mathbb{R})) \arrow[r, "\beta_1"] \arrow[d, "\phi"] & H^2(X; T) \arrow[r, "\overset{\alpha}{\cong}", phantom] \arrow[d, equal] & H^3(X; \mathbb{Z}) \arrow[d, equal] \\
H^1(X; T) \arrow[r] & H^1(X; U(H)) \arrow[r] & H^1(X; PU(H)) \arrow[r, "\beta_2"'] & H^2(X; T) \arrow[r, "\underset{\alpha}{\cong}", phantom] & H^3(X; \mathbb{Z})
\end{tikzcd}
\tag{2.1}
$$

Consider $x \in H^1(X; \mathrm{Sp}(n, \mathbb{R}))$. We have

$$\beta_1(x) = \beta_2(\phi(x)).$$

If x is a 1-cocycle which represents (F, ω), then, by (1.8), $\phi(x)$ is a 1-cocycle which represents $W(F, \omega)$. But $\alpha\beta_2 = \delta$, the Dixmier–Douady map, and so

$$\alpha\beta_1(x) = 0 \Leftrightarrow \delta(\phi(x)) = 0.$$

That is, the Dixmier–Douady invariant of the Weyl bundle vanishes if and only if the principal $\mathrm{Sp}(n, \mathbb{R})$)-bundle P determined by (F, ω) lifts to a principal G-bundle. Now the unitary group $U(n)$ is a maximal compact subgroup of $\mathrm{Sp}(n, \mathbb{R})$. We reduce the structure group of P from $\mathrm{Sp}(n, \mathbb{R})$ to $U(n)$. Then P lifts if the inclusion $U(n) \to \mathrm{Sp}(n, \mathbb{R})$ lifts to a continuous homomorphism l:

$$
\begin{tikzcd}
 & G \arrow[d] \\
U(n) \arrow[r] \arrow[ur, dashed, "l"] & \mathrm{Sp}(n, \mathbb{R})
\end{tikzcd}
$$

We proceed to prove this, and to write down l explicitly.

The inclusion $U(n) \to \mathrm{Sp}(n, \mathbb{R})$ is given by

$$A + iB \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix}.$$

Let $MU(n)$, the meta-unitary group, be the pre-image of $U(n)$ under the covering map $\mathrm{Mp}(n, \mathbb{R}) \to \mathrm{Sp}(n, \mathbb{R})$. We have a short exact sequence

$$1 \to \{e_0, e_1\} \to MU(n) \to U(n) \to 1,$$

where $\{e_0, e_1\}$ is the kernel of the covering map $MU(n) \to U(n)$ and e_0 is the identity element of $MU(n)$. There is a natural map

$$\pi_1(U(n)) \to \{e_0, e_1\} \cong \mathbb{Z}_2$$

induced by

$$\sigma \to \tilde{\sigma}(1),$$

where $\tilde{\sigma}$ is the unique lift to $MU(n)$ of the loop σ: $[0, 1] \to U(n)$. Set $\gamma(t) = \det(\sigma(t))$ for $0 \leqslant t \leqslant 1$. There is a natural map

$$\pi_1(U(1)) \to \{-1, +1\} \cong \mathbb{Z}_2$$

induced by

$$\gamma \longmapsto \tilde{\gamma}(1),$$

where $\tilde{\gamma}$ is the unique lift to $\mathrm{Spin}(2) \cong U(1)$ of the loop γ in $SO(2) \cong U(1)$. Now the induced map $\det_*: \pi_1(U(n)) \to \pi_1(U(1))$ is an isomorphism. Therefore

$$\begin{aligned} \tilde{\sigma}(1) = e_0 &\Leftrightarrow \tilde{\gamma}(1) = +1, \\ \tilde{\sigma}(1) = e_1 &\Leftrightarrow \tilde{\gamma}(1) = -1. \end{aligned} \tag{2.2}$$

Now let $y \in U(n)$ and let τ be a path in $U(n)$ such that $\tau(0) = I$ and $\tau(1) = y$. Let $\delta(t) = \det(\tau(t))$ for $0 \leqslant t \leqslant 1$. Let $\tilde{\tau}$ be the unique lift of τ to $MU(n)$, and let $\tilde{\delta}$ be the unique lift of δ to $\mathrm{Spin}(2) \cong U(1)$. Consider another path τ' such that $\tau'(0) = I$ and $\tau'(1) = y$. Then τ and τ' differ by a loop in $U(n)$. It follows immediately from (2.2) that

$$(\tilde{\tau}(1), \tilde{\delta}(1)) = (\tilde{\tau}'(1), \tilde{\delta}'(1))$$

or

$$(\tilde{\tau}(1), \tilde{\delta}(1)) = (e_1 \tilde{\tau}'(1), -\tilde{\delta}'(1)). \tag{2.3}$$

The argument in Weil [7, p. 196] shows that

$$G \cong (\mathrm{Mp}(n, \mathbb{R}) \times U(1))/\mathbb{Z}_2,$$

where the non-trivial element ε in $\mathbb{Z}_2$ acts on $x \in \mathrm{Mp}(n, \mathbb{R})$ as $\varepsilon \cdot x = e_1 x$, and on $z \in U(1)$ as $\varepsilon \cdot z = -z$. Let $[x, z]$ be the $\mathbb{Z}_2$-coset of (x, z). It follows from (2.3) that

$$[\tilde{\tau}(1), \tilde{\delta}(1)] \in \mathrm{Mp}(n, \mathbb{R}) \times U(1))/\mathbb{Z}_2$$

is independent of the path τ. Accordingly, we define

$$l(y) = [\tilde{\tau}(1), \tilde{\delta}(1)], \qquad y \in U(n). \tag{2.4}$$

Then l is a homomorphism because $U(n)$ is connected and $l(xy) = l(x)\, l(y)$ for x and y close enough to I.

Let $\mathsf{u}(n)$, $\mathsf{sp}(n, \mathbb{R})$, $\mathsf{mp}(n, \mathbb{R})$ be the Lie algebras of $U(n)$, $\mathrm{Sp}(n, \mathbb{R})$, $\mathrm{Mp}(n, \mathbb{R})$ so that $dj\colon \mathsf{mp}(n, \mathbb{R}) \cong \mathsf{sp}(n, \mathbb{R})$. Let the exponential maps be denoted by

$$\exp\colon \mathsf{sp}(n, \mathbb{R}) \to \mathrm{Sp}(n, \mathbb{R}),$$
$$\mathrm{Exp}\colon \mathsf{mp}(n, \mathbb{R}) \to \mathrm{Mp}(n, \mathbb{R}),$$

and let $e(x) = e^{ix}$ for $x \in \mathbb{R}$. Let $y \in U(n)$. There exists $v \in \mathsf{u}(n)$ such that $\exp(v) = y$. Then

$$\tau(t) = \exp tv, \qquad 0 \leqslant t \leqslant 1,$$

is a path such that $\tau(0) = I$ and $\tau(1) = y$. Then the map l is given explicitly, according to (2.4), as

$$l(\exp v) = [\mathrm{Exp}\, v, e(\mathrm{Tr}\, v/2)], \qquad v \in \mathsf{u}(n). \tag{2.5}$$

There is a short exact sequence

$$1 \to \mathrm{Mp}(n, \mathbb{R}) \to G \to T \to 1, \tag{2.6}$$

where $\mathrm{Mp}(n, \mathbb{R}) \to G$ sends x to $[x, 1]$ and $G \to T$ sends $[x, z]$ to z^2. When the trace of v is 0, so that $\exp v \in SU(n)$, we therefore have

$$l(\exp v) \in \mathrm{Mp}(n, \mathbb{R})$$

by (2.5) and (2.6). The lifting

$$\begin{tikzcd} & \mathrm{Mp}(n, \mathbb{R}) \arrow[d] \\ SU(n) \arrow[r] \arrow[ur] & \mathrm{Sp}(n, \mathbb{R}) \end{tikzcd}$$

is denoted $\tilde{\psi}$ in Borel and Wallach [1, p. 240].

3. Proof of Theorem 2

The Weyl bundle has Dixmier–Douady invariant 0: consequently there exists an underlying Hilbert bundle S. This bundle may be called, following Kostant [5], a bundle of symplectic spinors.

Let $G = \mathrm{Mp}(n, \mathbb{R}) \times_{\mathbb{Z}_2} U(1)$. The notation $G = \mathrm{Mp}^c(n, \mathbb{R})$ commends itself, by analogy with the notation $\mathrm{Spin}^c(n) = \mathrm{Spin}(n) \times_{\mathbb{Z}_2} U(1)$.

Definition. An Mp^c-*structure* on a symplectic bundle (F, ω) is a pair (η, β), where

(i) η is a principal $\mathrm{Mp}^c(n, \mathbb{R})$-bundle over X; and

(ii) β is a definite isomorphism $\eta \times_{\mathrm{Mp}^c(n,\mathbb{R})} \mathbb{R}^{2n} \cong F$.

Definition. A Hilbert bundle S over X is an *irreducible Weyl module* if there is a definite isomorphism θ: $W(F, \omega) \cong LC(S)$.

Here, $LC(S)$ is the bundle of elementary C^*-algebras determined by S: the fibre at x is $LC(S_x)$, the C^*-algebra of compact operators on S_x. The Weyl module structure on S is given by the bilinear map $W(F, \omega) \times S \to S$ sending u, s to $\theta(u)\, s$.

Exactly as in [4], there is a canonical bijection of the set of Mp^c-structures on (F, ω) with the set of irreducible $W(F, \omega)$-modules. This bijection is as follows.

An Mp^c-structure (η, β) on (F, ω) determines a U-structure on $W(F, \omega)$ according to the pull-back square in the category of topological groups:

$$\begin{array}{ccc} \mathrm{Mp}^c(n, \mathbb{R}) & \longrightarrow & U(H) \\ \downarrow & & \downarrow \\ \mathrm{Sp}(n, \mathbb{R}) & \longrightarrow & PU(H) \end{array}$$

The U-structure is a principal $U(H)$-bundle ξ and a definite isomorphism α: $\xi \times_{U(H)} LC(H) \cong W(F, \omega)$. Define

$$S = \xi \underset{U(H)}{\times} H,$$

$$\theta: LC(S) = \xi \underset{U(H)}{\times} LC(H) \overset{\alpha}{\cong} W(F, \omega).$$

Then S is an irreducible Weyl module.

The bundle of symplectic spinors always exists and, as a Hilbert bundle, is necessarily trivial. However, S may admit several irreducible-Weyl-module

structures. As in [4], the group $H^2(X; \mathbb{Z})$ acts simply-transitively on the set of irreducible Weyl modules S. The right action of $H^2(X; \mathbb{Z})$ is

$$S \cdot x = S \otimes L(x),$$

where $L(x)$ is the complex Hermitian line bundle whose first Chern class is x.

A symplectic vector bundle (F, ω) determines a cohomology class $c = c(F, \omega) \in H^2(X; \mathbb{Z})$.

Following Kostant [5, p. 145] one may define c as follows. The bundle P of symplectic frames is a principal $\mathrm{Sp}(n, \mathbb{R})$-bundle. Now the unitary group $U(n)$ is a maximal compact subgroup of $\mathrm{Sp}(n, \mathbb{R})$. We reduce the structure group from $\mathrm{Sp}(n, \mathbb{R})$ to $U(n)$, thus obtaining a principle $U(n)$-bundle. The class c is the first Chern class of this principal $U(n)$-bundle. Equivalently, c is the integral class of the corresponding determinant bundle N, which is a complex Hermitian line bundle.

Now G admits a unique unitary character $\phi: G \to U(1)$ such that $\phi(zI) = z^2$ for all z in $U(1)$. We call ϕ the *Weil character* [7, p. 196]. Its kernel is $\mathrm{Mp}(n, \mathbb{R})$ and ϕ is given by

$$[x, z] \to z^2, \qquad x \in \mathrm{Mp}(n, \mathbb{R}), \quad z \in U(1).$$

It now follows from the lifting formula (2.5) that the following diagram is commutative:

$$\begin{tikzcd} U(1) & G \arrow[l, "\phi"'] \arrow[d, "r"] \\ U(n) \arrow[u, "\det"] \arrow[ur, "l"] \arrow[r] & \mathrm{Sp}(n, \mathbb{R}) \end{tikzcd} \tag{3.1}$$

Now P lifts to a principal G-bundle $\tilde{P}$. The Weil character then determines an *associated* complex Hermitian line bundle λ. The square (3.1) proves that

$$\lambda = N. \tag{3.2}$$

The exact coefficient sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2 \to 0$ determines a homomorphism

$$\rho: H^2(X; \mathbb{Z}) \to H^2(X; \mathbb{Z}/2)$$

which is mod 2 reduction.

Let κ be the characteristic class of the Weyl bundle as in [4]. Thus $\kappa=\rho(c_1(\mu))$, where

$$\mu=\mathrm{Hom}_{W(F,\omega)}(S,S^*)$$

and S^* is the dual of S.

A reduction of structure group from $\mathrm{Sp}(n,\mathbb{R})$ to $U(n)$ determines a complex Hermitian vector bundle (E,b). Upon restriction of scalars, E becomes F; and the imaginary part of the inner product b is ω. Then

$$\begin{aligned}\kappa&=\rho(c_1(\mu)) && \text{by definition}\\ &=\rho(c_1(\lambda)) && \text{as in [4]}\\ &=\rho(c_1(N)) && \text{by (3.2)}\\ &=\rho(c_1(E))\\ &=w_2(F)\end{aligned}$$

since the mod 2 reduction of the first Chern class of a complex vector bundle is the second Stiefel–Whitney class of the underlying real vector bundle: whence Theorem 2.

Let $\mathfrak{A}$ be the C^*-algebra of sections of $W(F,\omega)$ which vanish at infinity. By Theorem 1, and the fact that the Hilbert bundle S is necessarily trivial, we have that $\mathfrak{A}$ is Morita equivalent to $C_0(X)$; so the K-theory of $\mathfrak{A}$ is the same as that of X: $K_*(\mathfrak{A})\cong K^*(X)$.

EXAMPLE. Let γ_n^1 be the canonical line bundle (Hopf bundle) over complex projective space $\mathbb{C}P^n$. Let (F,ω) be the associated symplectic bundle, so that F is a real 2-plane bundle. We have $H^2(\mathbb{C}P^n;\mathbb{Z})=\mathbb{Z}$ so there are $\aleph_0$ irreducible Weyl modules. The coefficient homomorphism $\rho\colon H^2(\mathbb{C}P^n;\mathbb{Z})\to H^2(\mathbb{C}P^n;\mathbb{Z}/2)$ sends $c_1(\gamma_n^1)$ to the generator of $H^2(\mathbb{C}P^n;\mathbb{Z}/2)=\mathbb{Z}/2$. But $\rho c_1(\gamma_n^1)=w_2(F)$. By Theorem 2, $\kappa(W(F,\omega))$ is the generator of $H^2(\mathbb{C}P^n;\mathbb{Z}/2)$.

ACKNOWLEDGMENTS

We would like to thank M. Takesaki for suggesting this project to us; and W. Arveson, J. Rawnsley and M. Takesaki for valuable conversations.

REFERENCES

1. A. BOREL AND N. WALLACH, "Continuous Cohomology, Discrete Subgroups, and Representations of Reductive Groups," Annals of mathematics study 94, Princeton, N. J., 1980.

2. J. Dixmier and A. Douady, Champs continus d'espaces hilbertiens et de C^*-algèbres, *Bull. Soc. Math. Fr.* **91** (1963), 227–284.
3. J. Frenkel, Cohomologie non abélienne et espaces fibrés, *Bull. Soc. Math. Fr.* **85** (1957), 135–220.
4. J. Hayden and R. J. Plymen, On the invariants of Serre and Dixmier–Douady, IHES preprint, 1981.
5. B. Kostant, Symplectic spinors, *Symp. Math.* **14** (1974), 139–152.
6. D. Kastler, The C^*-algebras of a free boson field, *Commun. Math. Phys.* **1** (1965), 14–48.
7. A. Weil, Sur certaines groupes d'opérateurs unitaires, *Acta Math.* **111** (1964), 143–211.
8. H. Weyl, Quantenmechanik und gruppentheorie, *Z. Phys.* **46** (1927), 1–46.