Available online at www.sciencedirect.com

SCIENCE DIRECT®

Advances in Mathematics 201 (2006) 516–548

ADVANCES IN Mathematics

www.elsevier.com/locate/aim

Moment conditions and support theorems for Radon transforms on affine Grassmann manifolds

Fulton B. Gonzalez[a,*], Tomoyuki Kakehi[b]

[a] *Department of Mathematics, Tufts University, Bromfield-Pearson Hall, Medford, MA 02155, USA*
[b] *Institute of Mathematics, University of Tsukuba, Tsukuba-shi,Ibaraki, 305-8571, Japan*

Received 30 July 2004; accepted 16 February 2005

Communicated by Gang Tian
Available online 10 May 2005

Abstract

Let $G(p,n)$ and $G(q,n)$ be the affine Grassmann manifolds of p- and q-planes in $\mathbb{R}^n$, respectively, and let $\mathcal{R}^{(p,q)}$ be the Radon transform from smooth functions on $G(p,n)$ to smooth functions on $G(q,n)$ arising from the inclusion incidence relation. When $p<q$ and $\dim G(p,n) = \dim G(p,n)$, we present a range characterization theorem for $\mathcal{R}^{(p,q)}$ via moment conditions. We then use this range result to prove a support theorem for $\mathcal{R}^{(p,q)}$. This complements a previous range characterization theorem for $\mathcal{R}^{(p,q)}$ via differential equations when $\dim G(p,n) < \dim G(p,n)$. We also present a support theorem in this latter case.

MSC: primary: 44A12; secondary: 43A85

Keywords: Radon transform; Grassmannian; Moment condition; Support theorem

1. Introduction

In this paper, we present a range characterization of Radon transforms on affine Grassmann manifolds via moment conditions. Our objective is to generalize the moment

* Corresponding author.
E-mail addresses: fulton.gonzalez@tufts.edu (F.B. Gonzalez), kakehi@math.tsukuba.ac.jp (T. Kakehi).

0001-8708/$ - see front matter
doi:10.1016/j.aim.2005.02.009

conditions for the classical Radon transform on $\mathbb{R}^n$ to affine Grassmannians; our results complement a previously obtained range characterization for these transforms using invariant differential equations.

Specifically, we consider the transform $\mathcal{R}^{(p,q)}$ from smooth functions on the space $G(p,n)$ of p-planes in $\mathbb{R}^n$ to smooth functions on the space $G(q,n)$ of q-planes in $\mathbb{R}^n$ arising from the inclusion incidence relation. Throughout this paper, we assume that $p < q$. A p-plane ℓ and a q-plane ξ are incident if $\ell \subset \xi$; the Radon transform is explicitly defined by

$$\mathcal{R}^{(p,q)} f(\xi) = \int_{\widehat{\xi}} f(\ell)\, d_\xi \ell. \tag{1.1}$$

when $\xi \in G(q,n)$ and f is a function on $G(p,n)$. Here $d_\xi \ell$ is a canonical measure on the set $\widehat{\xi} = \{ \ell \in G(p,n) \mid \ell \text{ is incident to } \xi \}$ invariant under all Euclidean motions preserving η. In particular, when $p = 0$, $\mathcal{R}^{(0,q)}$ reduces to the classical q-plane transform on $\mathbb{R}^n$. We will be working in the category $\mathcal{S}(G(p,n))$ of Schwartz class functions on $G(p,n)$, as defined in [Ri]. The following theorem [GK1, Theorem 7.7] gives a range characterization of $\mathcal{R}^{(p,q)}$ in the case when $\dim G(p,n) < \dim G(q,n)$:

Theorem 1.1. *Assume that* $p < q$ *and* $\dim G(p,n) < \dim G(q,n)$. *Then there exists a differential operator* Ω *of order* $2p+4$, *invariant under the Euclidean motion group* $M(n)$, *such that for any* $\varphi \in \mathcal{S}(G(q,n))$,

$$\varphi \in \mathcal{R}^{(p,q)}\mathcal{S}(G(p,n)) \iff \Omega\varphi = 0. \tag{1.2}$$

This theorem generalizes the range characterization of the k-plane transform on $\mathbb{R}^n$, when $k < n-1$, via a second-order ultrahyperbolic system [Ri,Go3] a single fourth-order $M(n)$-invariant differential operator [Go2].

The classical Radon transform, on the other hand, has its range given by moment conditions, as specified in [GGV,Lu,H1]. (See the last reference for a complete proof.) The Grassmannian analogue to this would correspond to the transform $\mathcal{R}^{(p,q)}$ with $\dim G(p,n) = \dim G(q,n)$. Since the dimensions coincide, we would expect the range to be specified by appropriate moment conditions as well. This is the content of our main result, Theorem 3.1 below. As with the proof in [H1], the crucial element of our proof consists of justifying the smoothness of a certain "partial Fourier transform" on the space $G(p,n)$. (Propositions 3.1 and 3.2 below.)

Now Helgason's geometric proof of the support theorem for the classical Radon transform is well-known ([H1]; less well-known is the fact that it is also a consequence of the forward (easy) moment conditions and a polar-coordinate version of the Paley–Wiener Theorem ([H3]; Theorem 2.3 below)). We give an extension of this theorem to affine Grassmannians (Theorem 6.1) and use it to prove a support theorem (Theorem 6.2) for the transform $\mathcal{R}^{(p,q)}$ in the case when $\dim(G(p,n)) = \dim(G(q,n))$. The injectivity of $\mathcal{R}^{(p,q)}$ is then used to prove the support theorem (Theorem 7.1) when $\dim(G(p,n)) < \dim(G(q,n))$.

2. Moment conditions and the support theorem for the classical Radon transform, revisited

To clarify our exposition, it will be instructive to briefly summarize Helgason's proof of the range characterization of the classical Radon transform by moment conditions. For this, let $\mathcal{R} = \mathcal{R}^{(0,n-1)} : f \mapsto \widehat{f}$ denote the classical Radon transform, which maps functions in $\mathcal{S}(\mathbb{R}^n)$ to functions $\varphi \in \mathcal{S}(\mathbf{S}^{n-1} \times \mathbb{R})$ which satisfy $\varphi(\omega, p) = \varphi(-\omega, -p)$. (See [H3, p. 99] for the definition of $\mathcal{S}(\mathbf{S}^{n-1} \times \mathbb{R})$.) We call such functions even and define $\mathcal{S}_H(\mathbf{S}^{n-1} \times \mathbb{R})$ to be the vector space of all even functions in $\mathcal{S}(\mathbf{S}^{n-1} \times \mathbb{R})$ satisfying the following condition:

For any $k \in \mathbb{Z}^+$, there exists a homogeneous polynomial P_k of degree k on $\mathbb{R}^n$ such that $\int_{-\infty}^{\infty} \varphi(\omega, p)\, p^k \, dp = P_k(\omega)$ for $\omega \in \mathbf{S}^{n-1}$. (H)

Theorem 2.1 (*Helgason [H1]*). $\mathcal{R}\mathcal{S}(\mathbb{R}^n) = \mathcal{S}_H(\mathbf{S}^{n-1} \times \mathbb{R})$.

Proof. It is an easy calculation to show that

$$\int_{-\infty}^{\infty} \mathcal{R}f(\omega, p)\, p^k \, dp = \int_{\mathbb{R}^n} f(x)\, \langle x, \omega \rangle^k \, dx.$$

This shows that $\mathcal{R}\mathcal{S}(\mathbb{R}^n) \subset \mathcal{S}_H(\mathbf{S}^{n-1} \times \mathbb{R})$. Conversely, suppose that $\varphi \in \mathcal{S}_H(\mathbf{S}^{n-1} \times \mathbb{R})$. Define the "partial Fourier transform" $\widetilde{\varphi}$ on $\mathbf{S}^{n-1} \times \mathbb{R}$,

$$\widetilde{\varphi}(\omega, s) = \int_{-\infty}^{\infty} \varphi(\omega, p)\, e^{-ips} \, dp, \qquad s \in \mathbb{R},\ \omega \in \mathbf{S}^{n-1}. \tag{2.1}$$

It is straightforward to prove that $\widetilde{\varphi}$ is an even function in $\mathcal{S}(\mathbf{S}^{n-1} \times \mathbb{R})$. In addition, the condition (H) for $k = 0$ shows that $\omega \mapsto \widetilde{\varphi}(\omega, 0)$ is constant so that there exists a unique function F on $\mathbb{R}^n$ for which $F(s\omega) = \widetilde{\varphi}(\omega, s)$. Now the map $(\omega, s) \mapsto s\omega$ is a local diffeomorphism of $\mathbf{S}^{n-1} \times (\mathbb{R} \setminus \{0\})$ onto $\mathbb{R}^n \setminus \{0\}$, which shows that F is smooth outside the origin. To prove the smoothness of F at the origin, it suffices to show that the partial derivatives of F are bounded on, say, the punctured unit ball $B' = \{ x \in \mathbb{R}^n \mid 0 < ||x|| < 1 \}$. Fix $\varepsilon > 0$. Now on the subset $A_{n,\varepsilon} = \{s\omega \in B' \mid 0 < s < 1,\ \omega = (\omega_1, \ldots, \omega_n) \in \mathbf{S}^{n-1},\ \omega_n > \varepsilon > 0\}$ we can use $(s, \omega_1, \ldots, \omega_{n-1})$ as local coordinates: repeated application of the chain rule shows that

$$\frac{\partial^k F}{\partial x_{i_1} \cdots \partial x_{i_k}}(x) = \sum_{j;i_1,\ldots,i_m} \frac{A_{j;i_1,\ldots,i_m}(\omega_1, \ldots, \omega_{n-1})}{s^{k-j}} \frac{\partial^m}{\partial \omega_{i_1} \cdots \partial \omega_{i_m}} \frac{\partial^j}{\partial s^j} F(s\omega). \tag{2.2}$$

The coefficients $A_{j;i_1,\ldots,i_m}(\omega_1, \ldots, \omega_{n-1})$ are smooth bounded functions of $(\omega_1, \ldots, \omega_{n-1})$ and the right-hand sum ranges over all $j \leqslant k$ and over all sequences $i_1, \ldots, i_m$ in $\{1, 2, \ldots, n-1\}$ where $m \leqslant k$.

We write $e^{-ips} = \sum_{l=0}^{k-1}(-ips)^l/l! + e_k(-ips)$ and apply condition (H) to (2.1) to obtain

$$F(s\omega) = \sum_{l=0}^{k-1} \frac{(-i)^l}{l!} P_l(s\omega) + \int_{-\infty}^{\infty} \varphi(\omega, p) e_k(-ips)\, dp.$$

Since the P_l are polynomials of degree l, (2.2) implies that

$$\frac{\partial^k F}{\partial x_{i_1} \cdots \partial x_{i_k}}(x) = \sum_{j;i_1,\ldots,i_m} A_{j;i_1,\ldots,i_m}(\omega) \int_{-\infty}^{\infty} \frac{\partial^m \varphi}{\partial \omega_{i_1} \cdots \partial \omega_{i_m}}(\omega, p) \times (-ip)^k \frac{e_{k-j}(-ips)}{(-ips)^{k-j}}\, dp. \tag{2.3}$$

Now $\varphi \in \mathcal{S}(S^{n-1} \times \mathbb{R})$ and $e_{k-j}(-it)/(-it)^{k-j}$ is bounded for all real t, so (2.3) shows that the kth-order derivatives of F are bounded on the set $A_{n,\varepsilon}$. From this, we deduce that the kth-order derivatives of F are bounded on B'. Hence $F \in C^\infty(\mathbb{R}^n)$. A routine calculation using, say, (2.2), shows that $F \in \mathcal{S}(\mathbb{R}^n)$. Denoting the Fourier transform on $\mathbb{R}^n$ by $\mathcal{F}$, let f be the inverse Fourier transform of F: $F = \mathcal{F}(f)$. For any ω, s the projection-slice theorem says that $\mathcal{F}f(s\omega) = \int_{-\infty}^{\infty} \widehat{f}(\omega, p)\, e^{-ips}\, dp$. The injectivity of the Fourier transform on $\mathbb{R}$ then implies that $\varphi = \widehat{f}$. $\square$

We now state the support theorem for the classical Radon transform in the following form:

Theorem 2.2 (*Helgason [H1]*). *Let $f \in \mathcal{S}(\mathbb{R}^n)$ and suppose that $R > 0$. If $\widehat{f}(\omega, p) = 0$ whenever $|p| > R$, then f has support in the closed ball $\bar{B}_R = \{x \in \mathbb{R}^n \mid \|x\| \leqslant R\}$.*

While the support theorem can be proved geometrically, it is also a consequence of the forward moment conditions (H). The key is the following polar coordinate version of the Paley–Wiener theorem. (See [H3, Exercise B1, Chapter 1].[1])

Theorem 2.3 (*Helgason [H2]*). *Let $R > 0$. The Fourier transform $f \mapsto \mathcal{F}f$ maps $\mathcal{D}(\bar{B}_R)$ onto the set of functions $\mathcal{F}f(\lambda\omega) = \psi(\omega, \lambda) \in C^\infty(S^{n-1} \times \mathbb{R})$ satisfying the following conditions*:
(i) *For each $\omega \in S^{n-1}$, the function $\lambda \mapsto \psi(\omega, \lambda)$ extends to a holomorphic function on $\mathbb{C}$ with the property that*

$$\sup_{\omega,\lambda} \left| \psi(\omega, \lambda)\, (1 + |\lambda|)^N\, e^{-R|Im\,\lambda|} \right| < \infty, \quad N \in \mathbb{Z}^+.$$

[1] The authors would like to thank Prof. S. Helgason for pointing out this exercise.

(ii) *For each* $k \in \mathbb{Z}^+$ *and each homogeneous degree* k *spherical harmonic function* h *on* S^{n-1}, *the function*

$$\lambda \mapsto \lambda^{-k} \int_{S^{n-1}} \psi(\omega, \lambda)\, h(\omega)\, d\omega$$

is even and holomorphic on $\mathbb{C}$ ($d\omega$ *denoting area measure on* S^{n-1}).

To see how the support theorem follows from this, we take $f \in \mathcal{S}(\mathbb{R}^n)$ satisfying $\widehat{f}(\omega, p) = 0$ for all $|p| > R$. Then $\psi(\omega, s) = \mathcal{F}f(s\omega)$ satisfies condition (i) above by the easy part of the Paley–Wiener theorem and the projection-slice theorem. Now let h be a homogeneous degree k spherical harmonic on S^{n-1}. Then

$$\begin{aligned}
\int_{S^{n-1}} \psi(\omega, s)\, h(\omega)\, d\omega &= \int_{S^{n-1}} \int_{-\infty}^{\infty} \widehat{f}(\omega, p)\, e^{-ips}\, dp\, h(\omega)\, d\omega \\
&= \sum_{l=0}^{k-1} \frac{1}{l!} \int_{S^{n-1}} \int_{-\infty}^{\infty} \widehat{f}(\omega, p)\, (-ips)^l\, dp\, h(\omega)\, d\omega \\
&\quad + \int_{S^{n-1}} \int_{-\infty}^{\infty} \widehat{f}(\omega, p)\, h(\omega)\, d\omega\, e_k(-ips)\, dp \\
&= \sum_{l=0}^{k-1} \frac{(-is)^l}{l!} \int_{S^{n-1}} P_l(\omega)\, h(\omega)\, d\omega \\
&\quad + \int_{S^{n-1}} \int_{-\infty}^{\infty} \widehat{f}(\omega, p)\, h(\omega)\, d\omega\, e_k(-ips)\, dp
\end{aligned}$$

by the forward moment conditions (H) for $\widehat{f}$. Since $P_l(\omega)$ is a sum of spherical harmonics of degree $\leqslant l$, the sum on the right-hand side vanishes, so that

$$s^{-k} \int_{S^{n-1}} \psi(\omega, s)\, h(\omega)\, d\omega = \int_{-\infty}^{\infty} s^{-k}\, e_k(-ips) \int_{S^{n-1}} \widehat{f}(\omega, p)\, h(\omega)\, d\omega\, dp.$$

The inner integral on the right is a smooth compactly supported function of $p \in \mathbb{R}$ and $s \mapsto s^{-k} e_k(-ips)$ extends to a holomorphic function on $\mathbb{C}$. In addition, the right-hand side above is easily seen to be even in s. Hence $\psi(\omega, s)$ satisfies condition (ii) in Theorem 2.3, and so f is supported in the closed ball $\bar{B}_R$.

3. The range of the Radon transform on affine Grassmannians: the equal rank case

We adopt the notation of [GK1] in what follows. Let $G_{p,n}$ be the (compact) Grassmann manifold of p-dimensional subspaces of $\mathbb{R}^n$. Then $G_{p,n} = O(n)/K_p$, where $K_p = O(p) \times O(n-p)$ is the subgroup of $O(n)$ fixing the p-plane $\sigma_0 = \mathbb{R}\mathbf{e}_1 \oplus \cdots \oplus \mathbb{R}\mathbf{e}_p$.

We assume that the Haar measure on all compact Lie groups, and the invariant measures on their homogeneous spaces (with the exception of the unit spheres), are normalized. Let $R_{p,q,n} : C^\infty(G_{p,n}) \to C^\infty(G_{q,n})$ be the Radon transform corresponding to the inclusion incidence relation between p- and q-dimensional subspaces of $\mathbb{R}^n$. Then $R_{p,q,n}$ is a linear bijection when $\operatorname{rank}(G_{p,n}) = \operatorname{rank}(G_{q,n})$ [Gr]; when $\operatorname{rank}(G_{p,n}) < \operatorname{rank}(G_{q,n})$, $R_{p,q,n}$ is injective and the range $R_{p,q,n}\, C^\infty(G_{p,n})$ is the subspace of $C^\infty(G_{q,n})$ annihilated by an $O(n)$-invariant differential operator of order $2\operatorname{rank}(G_{p,n}) + 2$ (See [K,GK1]).

When $\operatorname{rank}(G_{p,n}) \leqslant \operatorname{rank}(G_{q,n})$, there is an $O(n)$-invariant operator $\Box_{p,q,n} : C^\infty(G_{p,n}) \to C^\infty(G_{p,n})$ which inverts $R_{p,q,n}$:

$$f = \Box_{p,q,n}\, R_{q,p,n} \circ R_{p,q,n}\, f, \quad f \in C^\infty(G_{p,n}). \tag{3.1}$$

The operator $\Box_{p,q,n}$, given explicitly in [K], corresponds to multiplication by a constant factor on each of the K-types in $L^2(G_{p,n})$, and is a differential operator when $q - p$ is even. We call $\Box_{p,q,n}$ a *reproducing operator*.

As stated in the introduction, we assume that $p < q$. Let η_0 denote the q-plane $\mathbb{R}\mathbf{e}_1 \oplus \cdots \oplus \mathbb{R}\mathbf{e}_q$, and let H_p and H_q denote the subgroups of the Euclidean motion group $M(n) = O(n) \rtimes \mathbb{R}^n$ fixing the p-plane σ_0 and the q-plane η_0, respectively. We have, in particular, $H_p = (O(p) \times O(n-p)) \rtimes \mathbb{R}^p$ and $G(p,n) = M(n)/H_p$.

For suitable compatible measures on $M(n)$, H_p, H_q, and $H_p \cap H_q$, the affine Grassmannian transform $\mathcal{R}^{(p,q)}$ is the Radon transform associated with the double fibration

$$\begin{array}{ccc} & M(n)/(H_p \cap H_q) & \\ \swarrow & & \searrow \\ G(p,n) = M(n)/H_p & & G(q,n) = M(n)/H_q \end{array}$$

The corresponding incidence relation between p- and q-planes in $\mathbb{R}^n$ is just inclusion.

We can write $\mathcal{R}^{(p,q)}$ explicitly in the following way [GK1]. $G(p,n)$ is a vector bundle over $G_{p,n}$ of rank $n-p$: its fiber over each $\sigma \in G_{p,n}$ is $\sigma^\perp$; each $\ell \in G(p,n)$ is written uniquely as $\ell = (\sigma, x)$, where σ is the parallel translate of ℓ through the origin and $\{x\} = \sigma^\perp \cap \ell$. We parametrize $G(q,n)$ in a similar manner: $G(q,n) = \{(\eta, v) \in G_{q,n} \times \mathbb{R}^n \mid v \perp \eta\}$. From [GK1] the transform $\mathcal{R}^{(p,q)}$ is then given by the formula

$$\mathcal{R}^{(p,q)} f(\eta, v) = \int_{\sigma \subset \eta} \left(\int_{\sigma^\perp \cap \eta} f(\sigma, v + x)\, dx \right) d\sigma \tag{3.2}$$

for any appropriate function f on $G(p,n)$. The outer integral is taken over the set $\{\sigma \in G_{p,n} \mid \sigma \subset \eta\}$, with respect to the normalized measure invariant under all $u \in O(n)$ preserving η.

Let $\mathcal{S}(G(p,n))$ and $\mathcal{S}(G(q,n))$ denote the spaces of Schwartz-class functions on $G(p.n)$ and $G(q,n)$, respectively. Then $\mathcal{R}^{(p,q)} : \mathcal{S}(G(p,n)) \to \mathcal{S}(G(q,n))$, by

Gonzalez and Kakehi [GK1, §6]. Next let $\mathcal{F}_p$ and $\mathcal{F}_q$ denote the *partial Fourier transform* (i.e. Fourier transform on the fibers) on $G(p,n)$ and $G(q,n)$, respectively:

$$\mathcal{F}_p f(\sigma, y) \;=\; \int_{\sigma^\perp} f(\sigma, x)\, e^{-i\langle y,x\rangle}\, dx, \qquad f \in \mathcal{S}(G(p,n)). \tag{3.3}$$

Using (3.2), it is not hard to prove the affine Grassmannian version of the *projection-slice theorem*:

$$\mathcal{F}_q \circ \mathcal{R}^{(p,q)} f(\eta, y) \;=\; \int_{\sigma\subset\eta} \mathcal{F}_p f(\sigma, y)\, d\sigma \tag{3.4}$$

for all $(\eta, y) \in G(q,n)$ [GK1, Proposition 6.1].

We define the *rank* of $G(p,n)$ to be $\min(p+1, n-p)$. Since $\dim G(p,n) = (p+1)(n-p)$, we note that $\operatorname{rank}(G(p,n)) \leqslant \operatorname{rank}(G(q,n))$ if and only if $\dim G(p,n) \leqslant \dim G(q,n)$. The range result Theorem 1.1 from [GK1] is essentially proved using the projection-slice theorem and the range characterization of the compact Radon transform $R_{p,q}$ on Grassmannians in $\mathbb{R}^{n-1}$ via $O(n-1)$-invariant differential operators. In addition, the projection-slice theorem and the inversion formula (3.1) for $R_{p,q}$ on $\mathbb{R}^{n-1}$ are used to prove an inversion formula [GK1, Theorem 6.4] for $\mathcal{R}^{(p,q)}$ when $\dim G(p,n) = \dim G(q,n)$ and $q-p$ is even. (See Rubin [Ru2] for another inversion formula for $\mathcal{R}^{(p,q)}$ without the parity restriction.)

It is not hard to obtain an analogue of the moment conditions (H) for the transform $\mathcal{R}^{(p,q)}$. Let $k \in \mathbb{Z}^+$ and $f \in \mathcal{S}(G(p,n))$. Then for any $(\eta, y) \in G(q,n)$ we have

$$\begin{aligned}
\int_{\eta^\perp} \mathcal{R}^{(p,q)} f(\eta, v)\, \langle v, y\rangle^k\, dv \;&=\; \int_{\eta^\perp} \langle v, y\rangle^k \int_{\sigma\subset\eta} \int_{\sigma^\perp\cap\eta} f(\sigma, v+x)\, dx\, d\sigma\, dv \\
&=\; \int_{\sigma\subset\eta} \int_{\eta^\perp} \int_{\sigma^\perp\cap\eta} f(\sigma, v+x)\, \langle v+x, y\rangle^k\, dx\, dv\, d\sigma \\
&=\; \int_{\sigma\subset\eta} \int_{\sigma^\perp} f(\sigma, w)\, \langle w, y\rangle^k\, dw\, d\sigma.
\end{aligned} \tag{3.5}$$

Here we have used the fact that $\sigma^\perp$ equals the orthogonal direct sum $(\sigma^\perp \cap \eta) \oplus \eta^\perp$. The inner integral above represents a smooth function $P_k(\sigma, y)$ on $G(p,n)$:

$$P_k(\sigma, y) \;=\; \int_{\sigma^\perp} f(\sigma, w)\, \langle w, y\rangle^k\, dw, \qquad y \in \sigma^\perp. \tag{3.6}$$

Clearly P_k is a homogeneous degree k polynomial on the fibers of $G(p,n)$.

Eqs. (3.5) and (3.6) lead us to define $\mathcal{S}_H(G(q,n))$ as the vector space consisting of all $\varphi \in \mathcal{S}(G(q,n))$ satisfying the following condition:

(H'): For each $k \in \mathbb{Z}^+$, there exists a C^∞ function P_k on $G(p,n)$ such that

1. *For any $\sigma \in G_{p,n}$, the function $y \mapsto P_k(\sigma, y)$ is a homogeneous polynomial of degree k on $\sigma^\perp$.*

2. *For all* $(\eta, y) \in G(q,n)$, *we have*

$$\int_{\eta^\perp} \varphi(\eta, v)\, \langle v, y\rangle^k \, dv \;=\; \int_{\sigma\subset\eta} P_k(\sigma, y)\, d\sigma. \tag{3.7}$$

Eq. (3.5) shows that the range $\mathcal{R}^{(p,q)}\mathcal{S}(G(p,n))$ is a subspace of $\mathcal{S}_H(G(q,n))$. Note that the condition (H') reduces to the classical condition (H) when $p = 0$ and $q = n-1$. We now state our main result:

Theorem 3.1. *Suppose that* $p < q$ *and* $\operatorname{rank}(G(p,n)) = \operatorname{rank}(G(q,n))$. *Then* $\mathcal{R}^{(p,q)}\mathcal{S}(G(p,n)) = \mathcal{S}_H(G(q,n))$.

Our proof roughly follows the lines of the classical proof. Let $\varphi \in \mathcal{S}_H(G(q,n))$. The partial Fourier transform $\widetilde{\varphi}$ of φ,

$$\widetilde{\varphi}(\eta, y) \;=\; \mathcal{F}_q\varphi(\eta, y) \;=\; \int_{\eta^\perp} \varphi(\eta, v)\, e^{-i\langle v, y\rangle}\, dv, \tag{3.8}$$

belongs to $\mathcal{S}(G(q,n))$ by Gonzalez and Kakehi [GK1]. We now introduce the "flag" manifold $F_{q,n} = \{(\eta, \omega) \in G_{q,n} \times S^{n-1} \mid \eta \perp \omega\}$. (Define $F_{p,n}$ similarly.) Then $\widetilde{\varphi}$ gives rise to a smooth function $\widetilde{\Phi}$ on $F_{q,n} \times \mathbb{R}$

$$\widetilde{\Phi}(\eta, \omega, r) \;=\; \widetilde{\varphi}(\eta, r\omega). \tag{3.9}$$

Note that $\widetilde{\Phi}(\eta, \omega, r) = \widetilde{\Phi}(\eta, -\omega, -r)$.

For each $\omega \in S^{n-1}$, let $G_p(\omega^\perp)$ and $G_q(\omega^\perp)$ denote the compact Grassmann manifolds of p- and q-dimensional subspaces of the $(n-1)$-dimensional space $\omega^\perp \subset \mathbb{R}^n$. Then $G_p(\omega^\perp)$ and $G_q(\omega^\perp)$ are diffeomorphic to $G_{p,n-1}$ and $G_{q,n-1}$, respectively, and are homogeneous spaces of the subgroup $O(\omega)$ of $O(n)$ fixing ω.

$F_{q,n}$ is a fiber bundle over S^{n-1} with fibers $G_p(\omega^\perp)$. If we identify the Grassmannian $G_{q,n-1}$ with $G_q(e_n^\perp)$, we can see that $F_{q,n}$ is the associated fiber bundle $O(n) \times_{O(n-1)} G_{q,n-1}$ of the principal bundle $O(n) \to S^{n-1}$, $\quad u \mapsto u \cdot e_n$. Let $\widetilde{\pi}_q : O(n) \times G_{q,n-1} \to F_{q,n}$, $(u, \eta) \mapsto (u\cdot\eta, u\cdot e_n)$ be the quotient map. Using local cross sections, it is easy to see that a function Φ is smooth on $F_{q,n}$ iff its lift $\Phi \circ \widetilde{\pi}_q$ is smooth on $O(n) \times G_{q,n-1}$.

For each $\omega \in S^{n-1}$, let $R^\omega_{p,q} : C^\infty(G_p(\omega^\perp)) \to C^\infty(G_q(\omega^\perp))$ be the $O(\omega)$-invariant Radon transform corresponding to the inclusion incidence relation, and let $R^\omega_{q,p}$ be the dual transform. $R^\omega_{p,q}$ is of course just a translate, under $O(n)$, of the transform $R_{p,q,n-1} : C^\infty(G_{p,n-1}) \to C^\infty(G_{q,n-1})$ defined in the beginning of this section. Since $p + q + 1 = n$, it follows that $\operatorname{rank}(G_p(\omega^\perp)) = \operatorname{rank}(G_q(\omega^\perp)) = \min(p, q)$, and so $R^\omega_{p,q}$ is a linear bijection. Let $\Box^\omega_{p,q}$ be the corresponding reproducing operator; the appropriate translate of (3.1) for $R^\omega_{p,q}$ is

$$\psi \;=\; \Box^\omega_{p,q} R^\omega_{q,p} \circ R^\omega_{p,q}\, \psi \tag{3.10}$$

for all $\psi \in C^\infty(G_p(\omega^\perp))$.

Let us now return to the function $\widetilde{\Phi}$ on $F_{q,n} \times \mathbb{R}$. Since $R^{\omega}_{p,q}$ is a bijection, there is, for each (ω, r), a unique smooth function $\widetilde{F}(\cdot, \omega; r)$ on G^{ω}_p such that

$$\widetilde{\Phi}(\eta, \omega; r) = \int_{\sigma \subset \eta} \widetilde{F}(\sigma, \omega; r)\, d\sigma \tag{3.11}$$

for all $(\eta, \omega; r) \in F_{p,n} \times \mathbb{R}$. We also express this as $\widetilde{\Phi}(\eta, \omega; r) = \left(R^{\omega}_{p,q} \widetilde{F}(\cdot, \omega; r)\right)(\eta)$. We can, of course, think of $\widetilde{F}$ as being a function on $F_{p,n} \times \mathbb{R}$.

We want to prove that $\widetilde{F}$ is smooth on $F_{p,n} \times \mathbb{R}$. (This is not completely obvious.) Since the variable r is fixed in (3.11), this reduces to showing that the map $(\sigma, \omega) \mapsto \widetilde{F}(\sigma, \omega; r)$ is C^∞ on $F_{p,n}$ for each r. In view of the inversion formula (3.10), let us consider the integral transform S, from functions on $F_{q,n}$ to functions on $F_{p,n}$, given by

$$S\Phi(\sigma, \omega) = \int_{\sigma \subset \eta \subset \omega^\perp} \Phi(\eta, \omega)\, d\eta = \left(R^{\omega}_{q,p} \Phi(\cdot, \omega)\right)(\sigma), \quad (\sigma, \omega) \in F_{p,n} \tag{3.12}$$

for all $\Phi \in C^\infty(F_{q,n})$. Here $d\eta$ denotes the canonical and normalized measure on the submanifold $\{\eta \in G_{q,n} \mid \sigma \subset \eta \subset \omega^\perp\}$ of $G_{q,n}$.

Lemma 3.1. *S is a continuous linear operator from $C^\infty(F_{q,n})$ to $C^\infty(F_{p,n})$.*

Proof. In fact S is the Radon transform associated with the double fibration

$$\begin{array}{ccc} & O(n)/(\widetilde{K}_q \cap \widetilde{K}_p) & \\ \swarrow & & \searrow \\ F_{q,n} = O(n)/\widetilde{K}_q & & F_{p,n} = O(n)/\widetilde{K}_p \end{array}$$

where $\widetilde{K}_q = O(q) \times O(n-q-1)$ and $\widetilde{K}_p = O(p) \times O(n-p-1)$ are the subgroups of $O(n)$ fixing $(\eta_0, e_n) \in F_{q,n}$ and $(\sigma_0, e_n) \in F_{p,n}$, respectively. Hence by Helgason [H3, Chapter I, Proposition 3.8], the transform S is a continuous linear operator from $C^\infty(F_{q,n})$ to $C^\infty(F_{p,n})$. $\square$

It will be useful to express S in terms of associated fiber bundles. In terms of the quotient maps $\widetilde{\pi}_q : O(n) \times G_{q,n-1} \to F_{q,n}$ and $\widetilde{\pi}_p : O(n) \times G_{p,n-1} \to F_{p,n}$, we have

$$S\Phi \circ \widetilde{\pi}_p(u, \sigma) = \int_{\substack{\eta \supset \sigma \\ \eta \perp e_n}} \Phi \circ \widetilde{\pi}_q(u, \eta)\, d\eta = \left(R_{q,p,n-1}(\Phi \circ \widetilde{\pi}_q)(u, \cdot)\right)(\sigma). \tag{3.13}$$

If Φ is smooth on $F_{q,n}$, then the right-hand side is smooth on $O(n) \times G_{p,n-1}$. (This also shows that $S\Phi$ is smooth on $F_{p,n}$ whenever Φ is smooth in $F_{q,n}$.)

Next we define the operator $\Box^{(p)}$ on $F_{p,n}$ by putting

$$\Box^{(p)} F(\sigma, \omega) = \Box^{\omega}_{p,q} F(\cdot, \omega)(\sigma) \tag{3.14}$$

for all $F \in C^\infty(F_{p,n})$. (In the above, the operator $\Box^{\omega}_{p,q}$ acts on the first argument.)

Lemma 3.2. *$\Box^{(p)}$ is a continuous linear operator on $C^\infty(F_{p,n})$.*

Proof. Again letting $\widetilde{\pi}_p : O(n) \times G_{p,n-1} \to F_{p,n}$ be the quotient map, the $O(n-1)$-invariance of $\Box_{p,q,n-1}$ implies that

$$(\Box^{(p)} G) \circ \widetilde{\pi}_p(u, \sigma) = \Box_{p,q,n-1} \left(G \circ \widetilde{\pi}_p\right)(u, \sigma) \quad \text{for } G \in C^\infty(F_{p,n}), \tag{3.15}$$

where $\Box_{p,q,n-1}$ acts on the second argument. Therefore, the continuity of $\Box^{(p)}$ follows from (3.15) and the continuity of $\Box_{p,q,n-1}$. $\Box$

In particular, $\Box^{(p)} F$ is a smooth function on $F_{p,n}$.

We now slightly modify the definitions of the operators S and $\Box^{(p)}$ in (3.12) and (3.14) so that they act on functions on $F_{q,n} \times \mathbb{R}$ and $F_{p,n} \times \mathbb{R}$, respectively. In other words we put

$$S\Psi(\sigma, \omega; r) = \int_{\sigma \subset \eta \subset \omega^\perp} \Psi(\eta, \omega; r)\, d\eta, \quad (\sigma, \omega) \in F_{p,n} \tag{3.16}$$

for all $\Psi \in C^\infty(F_{q,n} \times \mathbb{R})$, and

$$\Box^{(p)} V(\sigma, \omega; r) = \Box^{\omega}_{p,q} V(\cdot, \omega; r)(\sigma) \tag{3.17}$$

for all $V \in C^\infty(F_{p,n} \times \mathbb{R})$. Lemmas 3.1 and 3.2, suitably modified, still apply to show that $S : C^\infty(F_{q,n} \times \mathbb{R}) \to C^\infty(F_{p,n} \times \mathbb{R})$ and $\Box^{(p)} : C^\infty(F_{p,n} \times \mathbb{R}) \to C^\infty(F_{p,n} \times \mathbb{R})$ are continuous linear operators.

Now by the inversion formula (3.10) for $R^{\omega}_{p,q}$ and definitions (3.12) and (3.14), we can recover $\widetilde{F}$ from $\widetilde{\Phi}$ in Eq. (3.11):

$$\widetilde{F}(\sigma, \omega; r) = \left(\Box^{(p)} \circ S\widetilde{\Phi}\right)(\sigma, \omega; r). \tag{3.18}$$

By the remarks above, we see that $\widetilde{F} \in C^\infty(F_{p,n} \times \mathbb{R})$.

The uniqueness of $\widetilde{F}$ in (3.11) implies that $\widetilde{F}(\sigma, \omega; r) = \widetilde{F}(\sigma, -\omega; -r)$ for all $(\sigma, \omega; r) \in F_{p,n} \times \mathbb{R}$. We next show that the moment conditions (H') for $k = 0$ imply that $\widetilde{F}(\sigma, \omega; 0)$ is constant in $\omega \in S^{n-1} \cap \sigma^\perp$. The function $P_0(\sigma, v)$ on $G(p, n)$ given in (H') in this case is a 0th-degree polynomial in v for each $\sigma \in G_{p,n}$, and thus depends

only on σ, so we put $P_0'(\sigma) = P_0(\sigma, 0)$. We have $P_0' \in C^\infty(G_{p,n})$ and by (3.7),

$$\widetilde{\phi}(\eta, 0) = \int_{\sigma\subset\eta} P_0'(\sigma)\, d\sigma.$$

If we take any $\omega \in S^{n-1}$ and $\eta \in G_q^\omega$, we have by (3.9) and (3.11),

$$\widetilde{\phi}(\eta, 0) = \widetilde{\Phi}(\eta, \omega, 0) = \int_{\sigma\subset\eta} \widetilde{F}(\sigma, \omega, 0)\, d\sigma.$$

Since $R_{p,q}^\omega$ is injective we see that $\widetilde{F}(\sigma, \omega, 0) = P_0'(\sigma)$ for all $\sigma \perp \omega$.

In view of this and the fact that $\widetilde{F}$ is even in (ω, r), there exists a function $\widetilde{f}$ on $G(p, n)$ given by

$$\widetilde{f}(\sigma, r\omega) = \widetilde{F}(\sigma, \omega; r), \quad (\sigma, \omega; r) \in F_{p,n} \times \mathbb{R}. \tag{3.19}$$

Now the mapping $(\sigma, \omega; r) \mapsto (\sigma, r\omega)$ is a local diffeomorphism from $F_{p,n} \times (\mathbb{R} \setminus \{0\})$ onto $G(p, n) \backslash G_{p,n}$. (This is best seen by viewing both $F_{p,n} \times \mathbb{R}$ and $G(p, n)$ as bundles over $G_{p,n}$, or as associated bundles of the principal bundle $O(n) \to G_{p,n},\ u \mapsto u \cdot \sigma_0$.)

$\widetilde{f}$ is therefore smooth on $G(p, n) \setminus G_{p,n}$. In addition, it is continuous on $G(p, n)$, since the map $(\sigma, \omega; r) \mapsto (\sigma, r\omega)$ is a quotient map of $F_{p,n} \times \mathbb{R}$ onto $G(p, n)$. From Eq. (3.11), $\widetilde{f}$ satisfies the relation

$$\widetilde{\phi}(\eta, y) = \int_{\sigma\subset\eta} \widetilde{f}(\sigma, y)\, d\sigma \tag{3.20}$$

for all $(\eta, y) \in G(q, n)$.

Our next objective, given in Propositions 3.1 and 3.2 below, is to prove that $\widetilde{f}$ is smooth on all of $G(p, n)$, and that in fact $\widetilde{f} \in \mathcal{S}(G(p, n))$. Assuming this, let f be the inverse partial Fourier transform of $\widetilde{f}$: $\mathcal{F}_p f = \widetilde{f}$. Then the projection-slice theorem 3.4, in conjunction with Eqs. (3.11), (3.8) (3.9), and (3.19) show that

$$\mathcal{F}_q \circ \mathcal{R}^{(p,q)} f(\eta, r\omega) = \int_{\sigma\subset\eta} \widetilde{f}(\sigma, r\omega)\, d\sigma = \mathcal{F}_q \varphi(\eta, r\omega)$$

for all $(\eta, \omega) \in F_{q,n}$ and all $r \in \mathbb{R}$. By the injectivity of $\mathcal{F}_q$, we get $\mathcal{R}^{(p,q)} f = \varphi$, which proves Theorem 3.1.

Proposition 3.1. $\widetilde{f} \in C^\infty(G(p, n))$.

Proposition 3.2. $\widetilde{f} \in \mathcal{S}(G(p, n))$.

We will give the proofs of the above two propositions in Section 5.

4. Differential operators on Grassmann manifolds and flag manifolds

In this section, we will study the calculus of differential operators on $F_{p,n}$ and on $G(p,n)\setminus(G_{p,n}\times\{0\})\cong F_{p,n}\times\mathbb{R}_+$. (Here we identify $G_{p,n}(\subset G(p,n))$ with $G_{p,n}\times\{0\}$, using the parametrization $G(p,n)\ni\ell=(\sigma,x)$, $\sigma\in G_{p,n}, x\in\sigma^{\perp}$.) In particular, we will give a kind of *polar coordinate decomposition* of differential operators on $G(p,n)\setminus(G_{p,n}\times\{0\})$ analogous to (2.2). The results in this section will be applied to the proofs of Propositions 3.1 and 3.2 in Section 5.

Let $M(n)=O(n)\rtimes\mathbb{R}^n$ be the Euclidean motion group, and let $\mathfrak{m}(n)=so(n)\rtimes\mathbb{R}^n$ be its Lie algebra. $\mathfrak{m}(n)$ has basis consisting of the elementary skew symmetric matrices $X_{ij}=E_{ij}-E_{ji}$ $(1\leqslant i<j\leqslant n)$ and E_k $(1\leqslant k\leqslant n)$, where $E_1,\ldots,E_n$ denote the infinitesimal translations in the directions of the unit vectors $\mathbf{e}_1,\ldots,\mathbf{e}_n$. Let $\mathfrak{u}(\mathfrak{m}(n))$ denote the universal enveloping algebra of $\mathfrak{m}(n)$. We note that $\mathfrak{u}(\mathbb{R}^n)\subset\mathfrak{u}(\mathfrak{m}(n))$ is a commutative subalgebra of $\mathfrak{u}(\mathfrak{m}(n))$. We call $P\in\mathfrak{u}(\mathbb{R}^n)$ a *homogeneous element* if P is written as a homogeneous polynomial of $E_1,\ldots,E_n$. By the Poincaré–Birkhoff–Witt theorem, the following proposition is easily obtained.

Proposition 4.1. *Any element $U\in\mathfrak{u}(\mathfrak{m}(n))$ is written as a linear combination of terms of the form QP, where $Q\in\mathfrak{u}(so(n))$ and $P\in\mathfrak{u}(\mathbb{R}^n)$ is a homogeneous element.*

For the sake of simplicity, throughout this section, we write the action of $X\in\mathfrak{m}(n)$ on $f\in C^\infty(G(p,n))$ as

$$X\cdot f((\sigma,x))=\frac{d}{dt}f(\exp(-tX)\cdot(\sigma,x))|_{t=0}.$$

Similarly, if $X\in so(n)$ and $f\in C^\infty(F_{p,n})$ (or $f\in C^\infty(G_{p,n})$), we write the action of X on f as $X\cdot f$. In addition, we also write the action of $U\in\mathfrak{u}(\mathfrak{m}(n))$ on $f\in C^\infty(G(p,n)$ as $U\cdot f$.

For $v\in\mathbf{S}^{n-1}$ and $\alpha>0$, we introduce the open sets

$$F^\alpha_{p,n}(v)\;=\;\{\,(\sigma,\omega)\in F_{p,n}\mid\|\mathrm{Pr}_{\sigma^\perp}v\|>\alpha\,\},$$

$\mathrm{Pr}_{\sigma^\perp}$ denoting orthogonal projection to $\sigma^\perp$. In addition, for an index set $I=\{i_1,\ldots,i_m\}$ with $1\leqslant i_1<\cdots<i_m\leqslant n$, let

$$F^\alpha_{p,n}[I]=\bigcap_{k=1}^m F^\alpha_{p,n}(\mathbf{e}_{i_k}).$$

Then we have the following:

Lemma 4.1.

$$\bigcup_{\#I=n-p}F^\alpha_{p,n}[I]\;=\;F_{p,n}\qquad\textit{for some }\ \alpha>0.$$

Proof. Take an arbitrary point $(\sigma, \omega) \in F_{p,n}$. Since σ is a p-dimensional subspace of $\mathbb{R}^n$, there exist $n-p$ unit vectors $\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_{n-p}}$ $(1 \leqslant i_1 < \cdots < i_{n-p} \leqslant n)$ such that $\mathbf{e}_{i_k} \notin \sigma$, $(1 \leqslant k \leqslant n-p)$. So if we put

$$\alpha = \frac{1}{2} \min\{ ||\mathrm{Pr}_{\sigma^\perp} \mathbf{e}_{i_k}|| \mid 1 \leqslant k \leqslant n-p \}, \quad I = \{ i_1, \ldots, i_{n-p} \},$$

then $(\sigma, \omega) \in F^{\alpha}_{p,n}[I]$. Let $V_\alpha = \bigcup_{\#I=n-p} F^{\alpha}_{p,n}[I]$. Then, the above result shows that $\bigcup_{\alpha>0} V_\alpha = F_{p,n}$. Since $\{ V_\alpha \}_{\alpha>0}$ is an open covering of the compact set $F_{p,n}$, we can take a finite covering $\{ V_{\alpha_j} \}_{j=1}^{N}$ of $F_{p,n}$. Let $\alpha_0 = \min_{1 \leqslant j \leqslant N} \alpha_j$. Then we have $V_{\alpha_0} = F_{p,n}$, which proves the assertion. $\square$

Lemma 4.2. *Let $\mathcal{U}_N = \{ \theta = (\theta_1, \ldots, \theta_N) \in \mathbf{S}^{N-1} \mid \theta_N > 0 \}$. There exist smooth functions a_j $(j = 1, \ldots, N-1)$ and b on $\mathbf{S}^{N-1} \times \mathcal{U}_N$ such that*

$$(E_v f)(r\theta) = \sum_{j=1}^{N-1} \frac{a_j(v,\theta)}{r} X_{jN} f(r\theta) + b(v,\theta) \frac{\partial}{\partial r} f(r\theta)$$

$$\text{for } (v,\theta) \in \mathbf{S}^{N-1} \times \mathcal{U}_N,\ r > 0 \quad \text{and for } f \in C^\infty(\mathbb{R}^N \setminus \{0\}).$$

Here E_v denotes the directional derivative in the direction of $-v \in \mathbb{R}^N$, namely,

$$E_v \cdot f(x) = \frac{d}{dt} f(x - tv)|_{t=0} \qquad \text{for } f \in C^\infty(\mathbb{R}^N).$$

Proof. It is not hard to see that

$$\frac{1}{r} X_{jN} = \theta_N \frac{\partial}{\partial x_j} - \theta_j \frac{\partial}{\partial x_N}, \quad \frac{\partial}{\partial r} = \sum_{j=1}^{N} \theta_j \frac{\partial}{\partial x_j}.$$

Hence

$$\begin{pmatrix} \frac{1}{r} X_{1N} \\ \vdots \\ \frac{1}{r} X_{N-1N} \\ \frac{\partial}{\partial r} \end{pmatrix} = \begin{pmatrix} \theta_N & 0 & \cdots & 0 & -\theta_1 \\ 0 & \theta_N & 0 & \cdots & -\theta_2 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & \theta_N & -\theta_{N-1} \\ \theta_1 & \theta_2 & \cdots & \theta_{N-1} & \theta_N \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_{N-1}} \\ \frac{\partial}{\partial x_N} \end{pmatrix}.$$

Note that

$$\det\begin{pmatrix} \theta_N & 0 & \cdots & 0 & -\theta_1 \\ 0 & \theta_N & 0 & \cdots & -\theta_2 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & \theta_N & -\theta_{N-1} \\ \theta_1 & \theta_2 & \cdots & \theta_{N-1} & \theta_N \end{pmatrix} = \theta_N^{N-2} > 0 \qquad \text{if } \theta \in \mathcal{U}_N.$$

Thus, each vector field $\frac{\partial}{\partial x_j}$ is written in the form

$$\frac{\partial}{\partial x_j} = \sum_{k=1}^{N-1} \frac{a_{jk}(\theta)}{\theta_N^{N-2}} \frac{1}{r} X_{kN} + \frac{b_j(\theta)}{\theta_N^{N-2}} \frac{\partial}{\partial r} \quad (j = 1, \ldots, N),$$

where $a_{jk}(\theta)$ and $b_j(\theta)$ are polynomials of $\theta \in \mathcal{U}_N$. Since $E_v = -\sum_{j=1}^{N} v_j \frac{\partial}{\partial x_j}$, the assertion follows from the above expression of $\frac{\partial}{\partial x_j}$. $\square$

Next, in a similar way to the case of $\mathbb{R}^N$ we introduce the radial derivative E_r on $G(p,n) \setminus (G_{p,n} \times \{0\})$ and the directional derivative E_v on $G(p,n)$ in the direction of $-v \in \mathbb{R}^n$ as follows:

$$E_r f(\sigma, x) = \frac{1}{||x||} \frac{d}{dr} f(\sigma, rx)|_{r=1} \qquad \text{for } f \in C^\infty(G(p,n) \setminus (G_{p,n} \times \{0\})),$$

$$E_v \cdot f(\ell) = \frac{d}{dt} f(\ell - tv)|_{t=0} \qquad \text{for } f \in C^\infty(G(p,n) \setminus (G_{p,n} \times \{0\})).$$

Our aim now is to generalize (2.2) to $G(p,n)$.

Proposition 4.2. *For an arbitrary point $(\sigma_1, \omega_1) \in F_{p,n}$, there exist an open neighborhood $\mathcal{U}$ of (σ_1, ω_1) in $F_{p,n}$ and smooth functions $a_{ij}^l(\sigma, \omega)$, $b^l(\sigma, \omega)$ on $\mathcal{U}$ such that*

$$E_l \cdot f(\sigma, r\omega) = \sum_{1 \leqslant i < j \leqslant n} \frac{1}{r} a_{ij}^l(\sigma, \omega)\, X_{ij} \cdot f(\sigma, r\omega) + b^l(\sigma, \omega)\, E_r f(\sigma, r\omega) \qquad (4.1)$$

for l $(1 \leqslant l \leqslant n)$, $(\sigma, \omega) \in \mathcal{U}$, $r > 0$ and for $f \in C^\infty(G(p,n) \setminus G_{p,n} \times \{0\})$.

Proof. Let us first take an open neighborhood $\mathcal{V}_1$ of σ_1 in $G_{p,n}$ and a smooth local cross section $u : \mathcal{V}_1 \to SO(n)$ such that

$$u(\sigma)\sigma = \mathbb{R}\mathbf{e}_1 \oplus \cdots \oplus \mathbb{R}\mathbf{e}_p \ \text{(we put } \equiv \sigma_0) \quad \text{for } \sigma \in \mathcal{V}_1, \quad u(\sigma_1)\omega_1 = \mathbf{e}_n.$$

Then, by taking a sufficiently small open neighborhood Ω_1 of ω_1 in $\mathbf{S}^{n-1}$, we have

$$u(\sigma)\omega \in \{ (0, \ldots, 0, \theta_1, \ldots, \theta_{n-p}) \in \mathbf{S}^{n-1} \mid \theta_{n-p} > 0 \}$$

for $(\sigma, \omega) \in (\mathcal{V}_1 \times \Omega_1) \cap F_{p,n}$. Now, let

$$\mathcal{U} = (\mathcal{V}_1 \times \Omega_1) \cap F_{p,n}.$$

By Lemma 4.1, there exists an index set $I = \{ i_1, \ldots, i_{n-p} \}$ such that $\mathcal{U} \subset F^{\alpha}_{p,n}[I]$. (Replace $\mathcal{V}_1$ and Ω_1 by smaller neighborhoods if necessary.)

Step 1: Let us first consider the case $l \in I$.

The vector $\mathbf{e}_l$ can be written as

$$\mathbf{e}_l = \mathrm{Pr}_{\sigma}\mathbf{e}_l + R_l(\sigma)\, y_l(\sigma),$$

$$R_l(\sigma) = ||\mathrm{Pr}_{\sigma^\perp}\mathbf{e}_l||, \quad y_l(\sigma) = \frac{\mathrm{Pr}_{\sigma^\perp}\mathbf{e}_l}{||\mathrm{Pr}_{\sigma^\perp}\mathbf{e}_l||} \in \mathbf{S}^{n-1} \cap \sigma^\perp. \tag{4.2}$$

By the definition of $F^{\alpha}_{p,n}[I]$,

$$R_l(\sigma) \equiv ||\mathrm{Pr}_{\sigma^\perp}\mathbf{e}_l|| > \alpha \quad \text{for } \sigma \in \mathcal{V}_1.$$

Therefore, $R_l(\sigma)$ and $y_l(\sigma)$ are smooth on $\mathcal{V}_1$. Moreover, by (4.2)

$$E_l \cdot f(\sigma, r\omega) = R_l(\sigma) E_{y_l(\sigma)} \cdot f(\sigma, r\omega) \quad \text{for } f \in C^\infty(G(p,n) \setminus G_{p,n} \times \{0\}).$$

From now on, we will decompose the above vector field $E_{y_l(\sigma)}$ as a linear combination of rotational derivatives and the radial derivative E_r. Since $u(\sigma)\cdot y_l(\sigma) \in \sigma_0^\perp = \mathbb{R}\mathbf{e}_{p+1} \oplus \cdots \oplus \mathbb{R}\mathbf{e}_n \cong \mathbb{R}^{n-p}$, we can apply Lemma 4.2 to the case when $N = n-p$ and $v = u(\sigma)\cdot y_l(\sigma)$. Thus

$$\begin{aligned} E_{u(\sigma)\cdot y_l(\sigma)} \cdot g(r\theta) = \sum_{m=p+1}^{n-1} \frac{1}{r} A_m(u(\sigma)\cdot y_l(\sigma), \theta)\, (X_{mn}\cdot g)(r\theta) \\ + B(u(\sigma)\cdot y_l(\sigma), \theta)\, (E_r\, g)(r\theta) \end{aligned} \tag{4.3}$$

for $\theta = (0, \ldots, 0, \theta_1, \ldots, \theta_{n-p}) \in \sigma_0^\perp \cap \mathbf{S}^{n-1}$, $\theta_{n-p} > 0$, $r > 0$, and for $g \in C^\infty(\mathbb{R}^{n-p} \setminus \{0\})$. Here A_m and B are smooth functions on $\mathbf{S}^{n-p-1} \times \{ \theta = (0, \ldots, 0, \theta_1, \ldots, \theta_{n-p}) \in \sigma_0^\perp \cap \mathbf{S}^{n-1} \mid \theta_{n-p} > 0 \}$. (We identify θ with a unit vector $(\theta_1, \ldots, \theta_{n-p})$ in $\mathbb{R}^{n-p}$.)

Let us take $g(r\theta) = f(u(\sigma)^{-1} \cdot \sigma_0,\ ru(\sigma)^{-1} \cdot \theta)$ in (4.3). Then,

$$\begin{aligned}
E_{y_l(\sigma)} \cdot f(\sigma, r\omega) &= E_{u(\sigma)\cdot y_l(\sigma)} \cdot f(u(\sigma)^{-1} \cdot \sigma_0,\ ru(\sigma)^{-1} \cdot \theta)|_{\theta=u(\sigma)\cdot\omega} \\
&= \sum_{m=p+1}^{n-1} \frac{1}{r} A_m(u(\sigma) \cdot y_l(\sigma), \theta) \\
&\quad \times X_{mn} \cdot f(u(\sigma)^{-1} \cdot \sigma_0, ru(\sigma)^{-1} \cdot \theta)|_{\theta=u(\sigma)\cdot\omega} \\
&\quad + B(u(\sigma) \cdot y_l(\sigma), \theta)\, E_r\, f(u(\sigma)^{-1} \cdot \sigma_0,\ ru(\sigma)^{-1} \cdot \theta)|_{\theta=u(\sigma)\cdot\omega} \\
&= \sum_{m=p+1}^{n-1} \frac{1}{r}\, A_m(u(\sigma) \cdot y_l(\sigma), \theta)\, (X_{mn}^{\tau(u(\sigma))} \cdot f)(\sigma, r\omega) \\
&\quad + B(u(\sigma) \cdot y_l(\sigma), \theta)\, (E_r^{\tau(u(\sigma))} f)(\sigma, r\omega).
\end{aligned} \tag{4.4}$$

We note that the radial derivative E_r is invariant under the action of $SO(n)$, in particular, $E_r^{\tau(u(\sigma))} = E_r$. In addition, we also note that $X_{mn}^{\tau(u(\sigma))}$ is written in the form

$$X_{mn}^{\tau(u(\sigma))} = \sum_{1 \leqslant i<j \leqslant n} C_{ij}^m(\sigma)\, X_{ij}, \tag{4.5}$$

where $C_{ij}^m(\sigma)$ is a smooth function on $\mathcal{V}_1$. Combining (4.4) and (4.5), we have the following expression:

$$E_{y_l(\sigma)} \cdot f(\sigma, r\omega) = \sum_{1 \leqslant i<j \leqslant n} \frac{1}{r} \widetilde{a}\,^l_{ij}(\sigma, \omega)\, X_{ij} \cdot f(\sigma, r\omega) + \widetilde{b}\,^l(\sigma, \omega)\, E_r f(\sigma, r\omega) \tag{4.6}$$

for some smooth functions $\widetilde{a}\,^l_{ij}$ and $\widetilde{b}\,^l$ on $\mathcal{U}$. Since $R_l(\sigma)$ is smooth on $\mathcal{V}_1$, we obtain an expression of form (4.1) for $l \in I$.

Step 2: Next, let us consider the case $l \notin I$. Since $\mathcal{U} \subset F_{p,n}^{\alpha}[I]$, we see easily that

$$\mathbb{R}^n = \sigma \oplus \langle \mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_{n-p}} \rangle \quad \text{for } \sigma \in \mathcal{V}_1, \tag{4.7}$$

where $\langle \mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_{n-p}} \rangle$ denotes the $(n-p)$-dimensional subspace spanned by $\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_{n-p}}$. (Note that decomposition (4.7) is not necessarily an orthogonal decomposition.) Using the decomposition (4.7), we can write $\mathbf{e}_l$ in the form

$$\mathbf{e}_l = z + \sum_{k=1}^{n-p} a_k(\sigma)\mathbf{e}_{i_k}, \quad z \in \sigma.$$

Obviously the coefficient $a_k(\sigma)$ is a smooth function of $\sigma \in \mathcal{V}_1$. Thus,

$$E_l \cdot f(\sigma, r\omega) = \sum_{k=1}^{n-p} a_k(\sigma) E_{i_k} \cdot f(\sigma, r\omega). \tag{4.8}$$

From (4.8) and the result of Step 1, we can conclude that $E_l f(\sigma, r\omega) = E_{\mathbf{e}_l} f(\sigma, r\omega)$ has an expression of the form (4.1) for $l \notin I$. $\square$

Proposition 4.2 yields the following:

Proposition 4.3. *There exist smooth functions $a^l_{ij}(\sigma, \omega)$ and $b^l(\sigma, \omega)$ on $F_{p,n}$ such that*

$$E_l \cdot f(\sigma, r\omega) = \sum_{1 \leqslant i < j \leqslant n} \frac{1}{r} a^l_{ij}(\sigma, \omega)\, X_{ij} \cdot f(\sigma, r\omega) + b^l(\sigma, \omega)\, E_r f(\sigma, r\omega) \tag{4.9}$$

for l $(1 \leqslant l \leqslant n)$, $(\sigma, \omega) \in F_{p,n}$, $r > 0$ and for $f \in C^\infty(G(p, n) \setminus G_{p,n} \times \{0\})$.

Proof. Since $F_{p,n}$ is compact, it follows from Proposition 4.2 that there exists a finite open covering $\{\mathcal{U}_v\}_{1 \leqslant v \leqslant m}$ of $F_{p,n}$ such that $E_l f$ has an expression of form (4.1) on each $\mathcal{U}_v$. So the above (globally defined) smooth functions $a^l_{ij}(\sigma, \omega)$ and $b^l(\sigma, \omega)$ can be constructed, using a partition of unity for the open covering $\{\mathcal{U}_v\}_{1 \leqslant v \leqslant m}$. $\square$

Let us consider $\mathfrak{u}(\mathbb{R}^n)$ to be a subalgebra of $\mathfrak{u}(\mathfrak{m}(n)) = \mathfrak{u}(\mathfrak{so}(n) \oplus \mathbb{R}^n)$. We see easily that an element of $\mathfrak{u}(\mathbb{R}^n)$ is written as a polynomial of vector fields $E_1, \ldots, E_n$. The following theorem is the affine Grassmann version of Eq. (2.2).

Theorem 4.1. *Let $P = P^{(m)}(E_1, \ldots, E_n) \in \mathfrak{u}(\mathbb{R}^n)$ be a homogeneous element of order m. Then for $f \in C^\infty(G(p, n) \setminus G_{p,n} \times \{0\})$, $P^{(m)}(E_1, \ldots, E_n) \cdot f(\sigma, r\omega)$ is expressed as*

$$\begin{aligned} &P^{(m)}(E_1, \ldots, E_n) \cdot f(\sigma, r\omega) \\ &\quad = \sum_{k=0}^{m} A^{(m)}_k(D_{(\sigma,\omega)}) r^{-k} E_r^{m-k} f(\sigma, r\omega) \quad \text{for } (\sigma, \omega) \in F_{p,n},\ r > 0, \end{aligned}$$

where $A^{(m)}_k(D_{(\sigma,\omega)})$ is a differential operator on $F_{p,n}$ of order at most k.

Proof. We will prove the theorem by induction with respect to m. In the case $m = 1$, the assertion follows from Proposition 4.3. Suppose that the assertion of the theorem holds for any element of $\mathfrak{u}(\mathbb{R}^n)$ of order at most m. Let us take any homogeneous element $Q \in \mathfrak{u}(\mathbb{R}^n)$ of order $m+1$. Then, Q can be written as a linear combination of $E_j Q^{(j)}$ $(1 \leqslant j \leqslant n)$, where $Q^{(j)} \in \mathfrak{u}(\mathbb{R}^n)$ is a homogeneous element of order m. Therefore, without loss of generality we may assume that $Q = E_1 P$ for some homogeneous

element $P \in \mathfrak{u}(\mathbb{R}^n)$ of order m. Then, by the hypothesis of induction, $Pf(\sigma, r\omega)$ is written as

$$P(E_1, \ldots, E_n)f(\sigma, r\omega) = \sum_{k=0}^{m} A_k(D_{(\sigma,\omega)})r^{-k}E_r^{m-k}f(\sigma, r\omega) \quad \text{for } (\sigma,\omega)\in F_{p,n},\ r>0,$$

where $A_k(D_{(\sigma,\omega)})$ is a differential operator on $F_{p,n}$ of order at most k.

On the other hand, by Proposition 4.3, there exist smooth functions $a_{ij}(\sigma, \omega)$ and $b(\sigma, \omega)$ on $F_{p,n}$ such that

$$E_1 \cdot F(\sigma, r\omega) = \sum_{1\leqslant i<j\leqslant n} \frac{1}{r} a_{ij}(\sigma,\omega)\, X_{ij} \cdot F(\sigma, r\omega) + b(\sigma,\omega)\, E_r F(\sigma, r\omega)$$

for $(\sigma, \omega) \in F_{p,n}$, $r > 0$ and for $F \in C^\infty(G(p,n) \setminus G_{p,n} \times \{0\})$. Here we note that

$$\begin{aligned} &rX_{ij} = X_{ij}r, \quad E_rX_{ij} = X_{ij}E_r, \quad rA_k(D_{(\sigma,\omega)}) = A_k(D_{(\sigma,\omega)})r, \\ &E_rA_k(D_{(\sigma,\omega)}) = A_k(D_{(\sigma,\omega)})E_r. \end{aligned}$$

Thus we have

$$\begin{aligned} Q \cdot f(\sigma, r\omega) &= E_1P \cdot f(\sigma, r\omega) \\ &= \sum_{1\leqslant i<j\leqslant n} \sum_{k=0}^{m} a_{ij}(\sigma,\omega)X_{ij}\, A_k(D_{(\sigma,\omega)})r^{-k-1}E_r^{m-k}\, f(\sigma, r\omega) \\ &\quad + \sum_{k=0}^{m} b(\sigma,\omega)\, A_k(D_{(\sigma,\omega)})\, r^{-k}E_r^{m+1-k}\, f(\sigma, r\omega) \\ &= b(\sigma,\omega)\, A_0(D_{(\sigma,\omega)})\, E_r^{m+1}\, f(\sigma, r\omega) \\ &\quad + \sum_{k=1}^{m} \left\{ b(\sigma,\omega)\, A_k(D_{(\sigma,\omega)}) + \sum_{1\leqslant i<j\leqslant n} a_{ij}(\sigma,\omega)X_{ij}\, A_{k-1}(D_{(\sigma,\omega)}) \right\} \\ &\quad \times r^{-k}E_r^{m+1-k}\, f(\sigma, r\omega) \\ &\quad + \sum_{1\leqslant i<j\leqslant n} a_{ij}(\sigma,\omega)X_{ij}\, A_m(D_{(\sigma,\omega)})\, r^{-m-1}\, f(\sigma, r\omega). \end{aligned}$$

It follows from the above expression that the assertion of the theorem holds for $Q = E_1P$. Therefore, the proof is completed. $\square$

5. Smoothness of $\widetilde{f}$

In this section, we will prove Propositions 3.1 and 3.2.

Proposition 5.1. *There exist homogeneous polynomials $P_k(\sigma, x)$ of degree k on $G(p,n)$ $(k = 0, 1, 2, \dots,)$ such that*

$$\widetilde{f}(\sigma, x) = \sum_{k=0}^{N-1} \frac{(-i)^k}{k!} P_k(\sigma, x) + (\Box^{(p)} \Phi_N)(\sigma, \omega; r),$$

$$\textit{where} \quad \Phi_N(\sigma, \omega; r) = \int_{\sigma \subset \eta \subset \omega^\perp} \int_{y \in \eta^\perp} e_N(-i\langle y, r\omega\rangle)\varphi(\eta, y)\, dy d\eta$$

for N, $(N = 1, 2, 3, \dots,)$ and for $(\sigma, x) \in G(p,n)$ with $x = r\omega$, $((\sigma, \omega) \in F_{p,n})$. Here, as in Section 2, $e_N(t)$ denotes the Nth remainder term of the Taylor expansion of e^t. Moreover, as in (3.12), *$d\eta$ denotes the canonical and normalized measure on the set $\{\eta \in G_{q,n} \mid \sigma \subset \eta \subset \omega^\perp\}$.*

Proof. By the definition of $\widetilde{F}$,

$$\begin{aligned}
\widetilde{f}(\sigma, r\omega) &= \Box^{\omega}_{p,q} \int_{\sigma \subset \eta \subset \omega^\perp} \mathcal{F}_q \varphi(\eta, r\omega)\, d\eta \\
&= \Box^{\omega}_{p,q} \int_{\sigma \subset \eta \subset \omega^\perp} \int_{y \in \eta^\perp} e^{-i\langle y, r\omega\rangle} \varphi(\eta, y)\, dy d\eta \\
&= \Box^{\omega}_{p,q} \int_{\sigma \subset \eta \subset \omega^\perp} \int_{y \in \eta^\perp} \left\{ \sum_{k=0}^{N-1} \frac{(-i)^k}{k!} \langle y, r\omega\rangle^k + e_N(-i\langle y, r\omega\rangle) \right\} \varphi(\eta, y)\, dy d\eta \\
&= \sum_{k=0}^{N-1} \frac{(-i)^k}{k!} \Box^{\omega}_{p,q} \int_{\sigma \subset \eta \subset \omega^\perp} \int_{y \in \eta^\perp} \varphi(\eta, y) \langle y, r\omega\rangle^k \, dy d\eta \\
&\quad + (\Box^{(p)} \Phi_N)(\sigma, \omega; r).
\end{aligned}$$

Since φ satisfies the moment condition (H'), there exist homogeneous polynomials $P_k(\sigma, x)$ of degree k on $G(p,n)$ $(k = 0, 1, 2, \dots)$, such that

$$\int_{\sigma \subset \eta} P_k(\sigma, r\omega)\, d\sigma = \int_{y \in \eta^\perp} \varphi(\eta, y) \langle y, r\omega\rangle^k \, dy \quad \text{for } \forall(\eta, r\omega) \in G(q,n) \text{ with } \eta \perp \omega.$$

Applying the inversion formula for $R^{\omega}_{p,q}$ to the both sides of the above equality, we have

$$P_k(\sigma, r\omega) = \Box^{\omega}_{p,q} \int_{\sigma \subset \eta \subset \omega^\perp} \int_{y \in \eta^\perp} \varphi(\eta, y) \langle y, r\omega\rangle^k \, dy d\eta.$$

Therefore, using these homogeneous polynomials P_k, $(k = 0, 1, 2, \ldots,)$, $\widetilde{f}(\sigma, r\omega)$ is written as

$$\widetilde{f}(\sigma, r\omega) = \sum_{k=0}^{N-1} \frac{(-i)^k}{k!} P_k(\sigma, r\omega) + (\square^{(p)} \Phi_N)(\sigma, \omega; r),$$

which completes the proof. $\square$

Since $G(p,n) \setminus (G_{p,n} \times \{0\}) \cong F_{p,n} \times \mathbb{R}_+$, the radial derivative E_r also acts on a function on $F_{p,n} \times \mathbb{R}_+$. From now on, we extend the action of E_r to $F_{p,n} \times (\mathbb{R} \setminus \{0\})$ so that

$$E_r \Psi(\sigma, \omega; r) = E_r \Psi(\sigma, -\omega; -r) \quad \text{for } \Psi \in C^\infty(F_{p,n} \times (\mathbb{R} \setminus \{0\})).$$

Proposition 5.2. *For $k, l \in \mathbb{Z}^+$ with $k + l = m$, we have*

$$(r^{-k} E_r^l \Phi_{m+1})(\cdot, \cdot; r) \to 0, \text{ as } r \to 0, \quad \text{in the topology of } C^\infty(F_{p,n}).$$

Proof. The Nth remainder term $e_N(t)$ in the Taylor expansion of e^t is expressed as

$$e_N(t) = t^N g_N(t), \quad g_N(t) = \frac{1}{(N-1)!} \int_0^1 (1-s)^N e^{st} \, ds. \tag{5.1}$$

Thus we have

$$\begin{aligned} \Phi_{m+1}(\sigma, \omega; r) &= \int_{\sigma \subset \eta \subset \omega^\perp} \int_{y \in \eta^\perp} (-i\langle y, r\omega\rangle)^{m+1} g_{m+1}(-i\langle y, r\omega\rangle) \varphi(\eta, y) \, dy d\eta \\ &= r^{m+1} \int_{\sigma \subset \eta \subset \omega^\perp} \int_{y \in \eta^\perp} (-i\langle y, \omega\rangle)^{m+1} \\ &\quad \times g_{m+1}(-i\langle y, r\omega\rangle) \varphi(\eta, y) \, dy d\eta. \end{aligned} \tag{5.2}$$

Let us write

$$r^{-k} E_r^l \left\{ r^{m+1} (-i\langle y, \omega\rangle)^{m+1} g_{m+1}(-i\langle y, r\omega\rangle) \right\} \varphi(\eta, y) = r \varphi_{m+1}(\eta, y; \omega, r). \tag{5.3}$$

Here we note that $\varphi_{m+1}(\eta, y; \omega, r)$ and its (higher order) derivatives with respect to y are bounded and integrable in y if $|r| \leqslant 1$. By (5.2) and (5.3), we have

$$\begin{aligned} &(r^{-k} E_r^l \Phi_{m+1})(u \cdot \sigma, u \cdot \omega; r) \\ &\quad = r \int_{\sigma \subset \eta \subset \omega^\perp} \int_{y \in \eta^\perp} \varphi_{m+1}(\eta, u \cdot y; \omega, r) \, dy d\eta \quad \text{for } u \in O(n). \end{aligned}$$

Thus we have for $X_1, \dots, X_L \in so(n)$

$$(X_1 \cdots X_L \cdot (r^{-k} E_r^l \,\Phi_{m+1}))(\sigma, \omega; r)$$
$$= r \int_{\sigma \subset \eta \subset \omega^\perp} \int_{y \in \eta^\perp} ((X_1)_y \cdots (X_L)_y \cdot \varphi_{m+1})(\eta, y; \omega, r)\, dy d\eta. \tag{5.4}$$

In equality (5.4), $(X_j)_y$, $(1 \leqslant j \leqslant L)$ acts on a function of y. Taking into account that $\varphi \in \mathcal{S}(G(q,n))$ and $g_{m+1}(-it)$ has bounded derivatives, we see easily that

$$\sup_{|r| \leqslant 1} |((X_1)_y \cdots (X_L)_y \cdot \varphi_{m+1})(\eta, y; \omega, r)| \leqslant C(1 + ||y||)^{-(n-q+1)} \quad \text{for } y \in \eta^\perp. \tag{5.5}$$

In inequality (5.5), the constant C does not depend on $(\eta, \omega) \in F_{q,n}$. As a result, for any $L \geqslant 1$ and for any $X_1, \dots, X_L \in so(n)$, there exists a constant C such that

$$\sup_{(\sigma,\omega) \in F_{p,n}} |(X_1 \cdots X_L \cdot (r^{-k} E_r^l \,\Phi_{m+1}))(\sigma, \omega; r)| \leqslant C|r| \qquad \text{for } r,\ (|r| < 1), \tag{5.6}$$

which proves the assertion. $\square$

Lemma 3.2 and Proposition 5.2 yield the following:

Proposition 5.3. *Let Q be an element of $\mathfrak{u}(so(n))$ and let $P = P^{(m)}(E_1, \dots, E_n)$ be a homogeneous element of $\mathfrak{u}(\mathbb{R}^n)$ of degree m. Then,*

$$\sup_{(\sigma,\omega) \in F_{p,n}} |(QP \cdot \square^{(p)} \Phi_{m+1})(\sigma, \omega; r)| \to 0 \quad \textit{as} \ r \to 0.$$

Proof. By Theorem 4.1, the differential operator $P^{(m)}(E_1, \dots, E_n)$ on $F_{p,n} \times (\mathbb{R} \setminus \{0\})$ is written in the form

$$P^{(m)}(E_1, \dots, E_n) = \sum_{k=0}^{m} A_k^{(m)}(D_{(\sigma,\omega)}) r^{-k} E_r^{m-k} \quad \text{for } (\sigma, \omega) \in F_{p,n},\ r \neq 0,$$

where $A_k^{(m)}(D_{(\sigma,\omega)})$ is a differential operator on $F_{p,n}$ of order at most k. Since $Q \in \mathfrak{u}(so(n))$, Q can be regarded as a differential operator on $F_{p,n}$. Moreover, $\square^{(p)}$ acts on functions of $(\sigma, \omega) \in F_{p,n}$ and therefore $\square^{(p)}$ commutes with the multiplication operator r and the radial derivative E_r. Taking account of these two facts, we have

$$(QP \cdot \square^{(p)} \Phi_{m+1})(\sigma, \omega; r) = \sum_{k=0}^{m} (\, Q \circ A_k^{(m)}(D_{(\sigma,\omega)}) \circ \square^{(p)} (r^{-k} E_r^{m-k} \Phi_{m+1})\,)(\sigma, \omega; r).$$

By Lemma 3.2, $\square^{(p)}$ is a continuous linear operator on $C^\infty(F_{p,n})$ and so is $Q \circ A_k^{(m)}(D_{(\sigma,\omega)}) \circ \square^{(p)}$. Thus, by Proposition 5.2, we have

$$\left(Q \circ A_k^{(m)}(D_{(\sigma,\omega)}) \circ \square^{(p)} \, (r^{-k} E_r^{m-k} \Phi_{m+1}) \right)(\cdot, \cdot; r) \to 0 \quad \text{as } r \to 0$$
$$\text{in the topology of } C^\infty(F_{p,n}).$$

In particular, we have

$$\sup_{(\sigma,\omega)\in F_{p,n}} \left| \left(Q \circ A_k^{(m)}(D_{(\sigma,\omega)}) \circ \square^{(p)} \, (r^{-k} E_r^{m-k} \Phi_{m+1}) \right)(\sigma, \omega; r) \right| \to 0 \quad \text{as } r \to 0,$$

which completes the proof. □

Proof of Proposition 3.1. From now on, we will prove the smoothness of $\widetilde{f}$ around the set $G_{p,n} \times \{0\}$. Since $\widetilde{f}$ is smooth on $G(p, n) \setminus (G_{p,n} \times \{0\})$, the proof is reduced to prove the following:

Proposition 5.4. *$\widetilde{f}$ is smooth near $G_{p,n} \times \{0\}$.*

Proof. It suffices to show that $\widetilde{f}$ is of class C^m for any m $(m = 0, 1, 2, \ldots,)$. We prove this by induction on m. We have already proved that $\widetilde{f}$ is continuous on $G(p, n)$, namely, $\widetilde{f}$ is of class C^0 on $G(p, n)$. In addition, we have already shown that $\widetilde{f}$ is smooth on $G(p, n) \setminus (G_{p,n} \times \{0\})$.

We assume that $\widetilde{f}$ is of class C^m on the set $\{(\sigma, x) \in G(p, n) \,|\, ||x|| < 1\}$. Let us take any $U \in \mathfrak{u}(\mathfrak{m}(n))$ of degree $m + 1$. From now on, we will prove that $U \cdot \widetilde{f}(\sigma, 0)$ exists and $U \cdot \widetilde{f}$ is continuous at $(\sigma, 0)$ for any point $(\sigma, 0) \in G_{p,n} \times \{0\}$.

Taking account of Proposition 4.1, we may assume that U is written in form (i) or (ii) below.

(i) $U = XQP$, where $X \in so(n)$, $Q \in \mathfrak{u}(so(n))$ is of degree k and $P \in \mathfrak{u}(\mathbb{R}^n)$ is a homogeneous element of degree $m - k$.
(ii) $U = E_1 P$, where $P \in \mathfrak{u}(\mathbb{R}^n)$ is a homogeneous element of degree m.

So if we prove the following two lemmas, Lemmas 5.1 and 5.2, then we see that $\widetilde{f}$ is of class C^{m+1}, and therefore, the proof of Proposition 5.4 is completed. □

Lemma 5.1. *Let $Q \in \mathfrak{u}(so(n))$ be of degree k and let $P \in \mathfrak{u}(\mathbb{R}^n)$ be a homogeneous element of degree l. We assume that $k + l = m$. Let $X \in so(n)$ and let $\sigma \in G_{p,n}$. Then, $QP \cdot \widetilde{f}$ is differentiable at $(\sigma, 0) \in G_{p,n} \times \{0\}$ with respect to X. Moreover, $XQP \cdot \widetilde{f}$ is continuous at $(\sigma, 0)$.*

Proof. By the hypothesis of the induction, $QP \cdot \widetilde{f}$ is a continuous function on $\{(\sigma, x) \in G(p, n) \,|\, ||x|| < 1\}$. By Proposition 5.1,

$$\widetilde{f}(\sigma, r\omega) = \sum_{j=0}^{l} \frac{(-i)^j}{j!} P_j(\sigma, x) + (\square^{(p)} \, \Phi_{l+1})(\sigma, \omega; r) \qquad \text{for } x = r\omega.$$

Since $P_j(\sigma, x)$ is a homogeneous polynomial of degree j and $P \in \mathfrak{u}(\mathbb{R}^n)$ is a homogeneous differential operator of order l, $P \cdot P_j(\sigma, x) = 0$ if $j \leqslant l - 1$. Thus we have

$$QP \cdot \widetilde{f}(\sigma, x) = \frac{(-i)^l}{l!}(QP \cdot P_l)(\sigma, x) + (QP \cdot \square^{(p)}\Phi_{l+1})(\sigma, \omega; r) \qquad \text{for } x = r\omega. \tag{5.7}$$

Let $r \to 0$ in (5.7). Then by Proposition 5.3 and the continuity of $QP \cdot \widetilde{f}$, we have

$$QP \cdot \widetilde{f}(\sigma, 0) = \frac{(-i)^l}{l!}(QP \cdot P_l)(\sigma, 0).$$

Hence,

$$\begin{aligned} X \cdot (QP \cdot \widetilde{f})(\sigma, 0) &= \frac{d}{dt}(QP \cdot \widetilde{f})(e^{-tX}\sigma, 0)|_{t=0} \\ &= \frac{(-i)^l}{l!}\frac{d}{dt}(QP \cdot P_l)(e^{-tX}\sigma, 0)|_{t=0} \\ &= \frac{(-i)^l}{l!}(XQP \cdot P_l)(\sigma, 0). \end{aligned}$$

Note that by the definition of moment condition $P_l \in C^\infty(G(p, n))$. Therefore, $QP \cdot \widetilde{f}$ is differentiable at $(\sigma, 0) \in G_{p,n} \times \{0\}$ with respect to X.

Next, we will show the continuity of $XQP \cdot \widetilde{f}$ at $(\sigma, 0)$. Similarly as in (5.7),

$$XQP \cdot \widetilde{f}(\sigma, x) = \frac{(-i)^l}{l!}(XQP \cdot P_l)(\sigma, x) + (XQP \cdot \square^{(p)}\Phi_{l+1})(\sigma, \omega; r) \qquad \text{for } x = r\omega.$$

Again, by Proposition 5.3,

$$\begin{aligned} \lim_{x\to 0}(XQP \cdot \widetilde{f})(\sigma, x) &= \frac{(-i)^l}{l!}\lim_{x\to 0}(XQP \cdot P_l)(\sigma, x) + \lim_{x\to 0}(XQP \cdot \square^{(p)}\Phi_{l+1})(\sigma, \omega; r) \\ &= \frac{(-i)^l}{l!}(XQP \cdot P_l)(\sigma, 0) = (XQP \cdot \widetilde{f})(\sigma, 0). \qquad \square \end{aligned}$$

Lemma 5.2. *Let $P \in \mathfrak{u}(\mathbb{R}^n)$ be a homogeneous element of degree m. Then, $P \cdot \widetilde{f}$ is differentiable at $(\sigma, 0) \in G_{p,n} \times \{0\}$ with respect to E_1. Moreover, $E_1 P \cdot \widetilde{f}$ is continuous at $(\sigma, 0)$.*

Proof. If $\mathbf{e}_1$ is parallel to σ, obviously the assertion holds. so we assume that $\mathbf{e}_1 \nparallel \sigma$. By the hypothesis of the induction, $P \cdot \widetilde{f}$ is continuous on $\{(\sigma, x) \in G(p, n) \mid ||x|| < 1\}$.

Let $v = \mathrm{Pr}_{\sigma^\perp}\mathbf{e}_1 \neq 0$. By the mean value theorem,

$$\frac{1}{t}\{P \cdot \widetilde{f}(\sigma, -tv) - P \cdot \widetilde{f}(\sigma, 0)\} = (E_v P \cdot \widetilde{f})(\sigma, -\theta t v) = (E_1 P \cdot \widetilde{f})(\sigma, -\theta t v) \tag{5.8}$$

for some θ, $(0 < \theta < 1)$.

By Proposition 5.1,

$$\widetilde{f}(\sigma, r\omega) = \sum_{j=0}^{m+1} \frac{(-i)^j}{j!} P_j(\sigma, x) + (\square^{(p)} \Phi_{m+2})(\sigma, \omega; r) \qquad \text{for } x = r\omega. \tag{5.9}$$

Since $P_j(\sigma, x)$ is a homogeneous polynomial of degree j and $E_1 P \in \mathfrak{u}(\mathbb{R}^n)$ is a homogeneous differential operator of order $m+1$, $E_1 P \cdot P_j(\sigma, x) = 0$ if $j \leqslant m$. Thus we have

$$\begin{aligned}(E_1 P \cdot \widetilde{f})(\sigma, r\omega) = {} & \frac{(-i)^{m+1}}{(m+1)!} (E_1 P \cdot P_{m+1})(\sigma, x) \\ & + (E_1 P \circ \square^{(p)} \Phi_{m+2})(\sigma, \omega; r) \qquad \text{for } x = r\omega.\end{aligned} \tag{5.10}$$

Hence by Proposition 5.3,

$$\begin{aligned}(E_1 P \cdot \widetilde{f})(\sigma, -\theta t v) &= \frac{(-i)^{m+1}}{(m+1)!} (E_1 P \cdot P_{m+1})(\sigma, -\theta t v) \\ &\quad + (E_1 P \circ \square^{(p)} \Phi_{m+2})(\sigma, v/||v||; -\theta t ||v||) \\ &\longrightarrow \frac{(-i)^{m+1}}{(m+1)!} (E_1 P \cdot P_{m+1})(\sigma, 0) \qquad \text{as } t \to 0.\end{aligned} \tag{5.11}$$

Therefore, it follows from (5.8) and (5.11) that $P \cdot \widetilde{f}$ is differentiable at $(\sigma, 0) \in G_{p,n} \times \{0\}$ with respect to E_1 and that

$$E_1 P \cdot \widetilde{f}(\sigma, 0) = \frac{(-i)^{m+1}}{(m+1)!} (P \cdot P_{m+1})(\sigma, 0). \tag{5.12}$$

Next we will show that $(E_1 P \cdot \widetilde{f})$ is continuous at $(\sigma, 0)$. By (5.10) and Proposition 5.3,

$$\begin{aligned}(E_1 P \cdot \widetilde{f})(\sigma, r\omega) &= \frac{(-i)^{m+1}}{(m+1)!} (E_1 P \cdot P_{m+1})(\sigma, x) \\ &\quad + (E_1 P \circ \square^{(p)} \Phi_{m+2})(\sigma, \omega; r)\end{aligned}$$

$$\begin{aligned} &\longrightarrow \frac{(-i)^{m+1}}{(m+1)!}(E_1 P \cdot P_{m+1})(\sigma, 0) \\ &= (E_1 P \cdot \widetilde{f})(\sigma, 0) \quad \text{as } r \to 0, \end{aligned} \tag{5.13}$$

which proves the continuity of $(E_1 P \cdot \widetilde{f})$ at $(\sigma, 0)$. $\square$

Finally we will prove that $\widetilde{f}$ is a Schwartz class function on $G(p, n)$.

Before going to the proof, let us recall from [GK1,Ri] that a smooth function g on $G(p, n)$ belongs to the Schwartz space $\mathcal{S}(G(p, n))$ if for any nonnegative integers N and m and for any m vector fields $Y_1, \ldots, Y_m \in \mathfrak{m}(n)$ g satisfies

$$\sup_{(\sigma,x)\in G(p,n)} \|x\|^N \, |(Y_1 \cdots Y_m \cdot g)(\sigma, x)| < \infty.$$

Proof of Proposition 3.2. We start with the following:

$$\widetilde{f}(\sigma, r\omega) = \square^{(p)} \int_{\sigma\subset\eta\subset\omega^\perp} \mathcal{F}_q\varphi(\eta, r\omega)\, d\eta.$$

Let us take any nonnegative integers m and N. In addition, let us take any element $Q \in \mathfrak{u}(so(n))$ and any homogeneous element $P = P^{(m)}(E_1, \ldots, E_n) \in \mathfrak{u}(\mathbb{R}^n)$ of degree m. By Theorem 4.1,

$$\begin{aligned} &r^N\, (QP^{(m)}(E_1, \ldots, E_n) \cdot \widetilde{f})(\sigma, r\omega) \\ &= r^N \sum_{k=0}^{m} QA_k^{(m)}(D_{(\sigma,\omega)}) r^{-k} E_r^{m-k}\, \square^{(p)} \int_{\sigma\subset\eta\subset\omega^\perp} \mathcal{F}_q\varphi(\eta, r\omega)\, d\eta \\ &= \sum_{k=0}^{m} QA_k^{(m)}(D_{(\sigma,\omega)})\square^{(p)} \int_{\sigma\subset\eta\subset\omega^\perp} r^{N-k} E_r^{m-k}\, \mathcal{F}_q\varphi(\eta, r\omega)\, d\eta \\ &= \sum_{k=0}^{m} (\, Q \circ A_k^{(m)}(D_{(\sigma,\omega)}) \circ \square^{(p)} \circ S\,)(\, r^{N-k} E_r^{m-k}\, \mathcal{F}_q\varphi(\cdot, r\cdot)\,)(\sigma, \omega), \end{aligned} \tag{5.14}$$

where $A_k^{(m)}(D_{(\sigma,\omega)})$ is a differential operator on $F_{p,n}$ of order at most k and where S is the Radon transform from $C^\infty(F_{q,n})$ to $C^\infty(F_{p,n})$ defined by (3.12). In the above computation, we used the fact that the multiplication operators r^j $(j = 1, 2, \ldots)$ commute with differential operators on $F_{p,n}$ and with the operator $\square^{(p)}$. Since $\mathcal{F}_q\varphi \in \mathcal{S}(G(q, n))$,

$$r^{N-k} E_r^{m-k}\, \mathcal{F}_q\varphi(\cdot, r\cdot) \longrightarrow 0 \qquad \text{as } |r| \to \infty,$$

in the topology of $C^\infty(F_{q,n})$. Moreover, in the summand of (5.14), $Q \circ A_k^{(m)}(D_{(\sigma,\omega)}) \circ \square^{(p)} \circ S$ is a continuous linear operator from $C^\infty(F_{q,n})$ to $C^\infty(F_{p,n})$. Thus we have

$$(Q \circ A_k^{(m)}(D_{(\sigma,\omega)}) \circ \square^{(p)} \circ S)(r^{N-k} E_r^{m-k} \mathcal{F}_q \varphi(\cdot, r\cdot)) \longrightarrow 0 \qquad \text{as } |r| \to \infty,$$

in the topology of $C^\infty(F_{p,n})$, from which we can conclude that

$$\sup_{(\sigma,\omega)\in F_{p,n}} |r^N (QP^{(m)}(E_1, \ldots, E_n) \cdot \widetilde{f})(\sigma, r\omega)| \longrightarrow 0 \qquad \text{as } |r| \to \infty. \tag{5.15}$$

Taking account of Proposition 4.1, we see that (5.15) proves the assertion. $\square$

6. Support Theorem 1

Our objective in this section is to prove a support theorem for $\mathcal{R}^{(p,q)}$ based on the forward moment conditions (H'), generalizing Theorem 2.2. For this, we make use of the second-order differential operator Δ_p on $G(p,n)$ which acts as the Laplacian on each fiber $\sigma^\perp$:

$$\Delta_p f(\sigma, x) = L_{\sigma^\perp} f(\sigma, x)$$

Δ_p is invariant under the motion group $M(n)$ [H1]. We define the differential operator Δ_q on $G(q,n)$ similarly.

Using the operator Δ_p, it makes sense to talk about harmonic polynomials on the fiber $\sigma^\perp$ in $G(p,n)$, as well as spherical harmonics on the unit sphere S_σ in $\sigma^\perp$. (These can also be obtained from harmonic polynomials on $\mathbb{R}^{n-p}$ by a translation by an appropriate element $u \in O(n)$.)

For any $R > 0$, define $\bar{\beta}_p^R$ to be the set of all p-planes (σ, x) at distance $\leqslant R$ from the origin. We will use the following analogue of Theorem 2.3 for the partial Fourier transform $\mathcal{F}_p$.

Theorem 6.1. *The partial Fourier transform $f \mapsto \mathcal{F}_p f$ maps $\mathcal{D}(\bar{\beta}_p^R)$ onto the space of all functions $\mathcal{F}_p f(\sigma, \lambda\omega) = \widetilde{F}(\sigma, \omega; \lambda) \in C^\infty(F_{p,n} \times \mathbb{R})$ satisfying the following conditions*:

(i) *For each $(\sigma, \omega) \in F_{p,n}$, the function $\lambda \mapsto \widetilde{F}(\sigma, \omega; \lambda)$ extends to a holomorphic function on $\mathbb{C}$ with the property that*

$$\sup_{(\sigma,\omega;\lambda)\in F_{p,n}\times\mathbb{C}} \left| (1 + |\lambda|)^N \widetilde{F}(\sigma, \omega; \lambda)\, e^{-R|\mathrm{Im}\lambda|} \right| < \infty \tag{6.1}$$

for each $N \in \mathbb{Z}^+$.

(ii) *For each $k \in \mathbb{Z}^+$, for each $\sigma \in G_{p,n}$, and for each homogeneous degree k harmonic polynomial h_σ on $\sigma^\perp$, the function*

$$\lambda \mapsto \lambda^{-k} \int_{S_\sigma} \widetilde{F}(\sigma, \omega; \lambda) h_\sigma(\omega)\, d\omega$$

is even and holomorphic in $\mathbb{C}$.

Proof. If $f \in \mathcal{D}(\bar{\beta}_p^R)$, then clearly the function $\lambda \mapsto \widetilde{F}(\sigma, \omega; \lambda) = \int_{\sigma^\perp} f(\sigma, x) e^{-i\lambda\langle x, \omega\rangle}\, dx$ extends to a holomorphic function on $\mathbb{C}$. If $N \in \mathbb{Z}^+$, we have

$$\begin{aligned}
|\lambda|^{2N} \left|\widetilde{F}(\sigma, \omega; \lambda)\right| &= \left|\lambda^{2N} \int_{\sigma^\perp} f(\sigma, x)\, e^{-i\lambda\langle x, \omega\rangle}\, dx\right| \\
&= \left|\int_{\sigma^\perp} (-\Delta_p)^N f(\sigma, x)\, e^{-i\lambda\langle x, \omega\rangle}\, dx\right| \\
&\leqslant C \max_{(\sigma, x) \in \bar{\beta}_p^R} |(-\Delta_p)^N f(\sigma, x)|\, e^{R|\mathrm{Im}\lambda|}
\end{aligned}$$

for all $(\sigma, \omega; \lambda) \in F_{p,n} \times \mathbb{C}$, which proves estimate (6.1). Moreover, $\widetilde{F}$ satisfies condition (ii) above by Theorem 2.3 applied to the Euclidean space $\sigma^\perp$.

Conversely, suppose $\mathcal{F}_p f = \widetilde{F}$ satisfies (i) and (ii). Let $\sigma \in G_{p,n}$. Then—again by Theorem 2.3 applied to the Euclidean space $\sigma^\perp$—the function

$$f(\sigma, x) = (2\pi)^{p-n} \int_{S_\sigma} \int_0^\infty \widetilde{F}(\sigma, \omega; \lambda)\, e^{i\lambda\langle x, \omega\rangle}\, \lambda^{n-p-1}\, d\lambda\, d\omega$$

satisfies $f(\sigma, x) = 0$ for all $||x|| > R$. This shows that f is supported in $\bar{\beta}_p^R$. □

Our main result, the support theorem below, extends the classical support theorem, Theorem 2.2.

Theorem 6.2 (*Support Theorem for $\mathcal{R}^{(p,q)}$*). *Assume that* $\mathrm{rank}(G(p,n)) = \mathrm{rank}(G(q,n))$. *Suppose that $f \in \mathcal{S}(G(p,n))$ satisfies $\mathcal{R}^{(p,q)} f(\eta, y) = 0$ whenever $|y| > R$. Then f is supported in $\bar{\beta}_p^R$.*

Proof. Let $\varphi = \mathcal{R}^{(p,q)} f$; then let $\widetilde{\Phi}(\eta, \omega; \lambda) = \mathcal{F}_q \varphi(\eta, \lambda\omega)$ as in (3.9). From Theorem 6.1, the function $\lambda \mapsto \widetilde{\Phi}(\eta, \omega; \lambda)$ extends to a holomorphic function on $\mathbb{C}$ for each $(\eta, \omega) \in F_{q,n}$, and satisfies the estimate

$$\sup_{(\eta, \omega; \lambda) \in F_{q,n} \times \mathbb{C}} \left|(1 + |\lambda|)^N\, \widetilde{\Phi}(\eta, \omega; \lambda)\, e^{-R|\mathrm{Im}\lambda|}\right| < \infty \qquad \text{for all } N \in \mathbb{Z}^+. \tag{6.2}$$

Moreover, since $X \cdot \mathcal{F}_q\varphi = \mathcal{F}_q(X \cdot \varphi)$ for $X \in so(n)$, we have a similar estimate for $X_1 \cdots X_m \cdot \widetilde{\Phi}$. Here $X_1, \dots, X_m \in so(n)$. Namely, for any nonnegative integer m and for any m vector fields $X_1, \dots, X_m \in so(n)$, we have

$$\sup_{(\eta,\omega;\lambda)\in F_{q,n}\times\mathbb{C}} \left|(1+|\lambda|)^N (X_1 \cdots X_m \cdot \widetilde{\Phi})(\eta,\omega;\lambda) e^{-R|\mathrm{Im}\lambda|}\right| < \infty \quad \text{for all } N \in \mathbb{Z}^+. \tag{6.3}$$

Let us fix an arbitrary nonnegative integer N and introduce a family of functions $\{H_\lambda \mid \lambda \in \mathbb{C}\}$ in $C^\infty(F_{q,n})$ as follows:

$$H_\lambda(\eta,\omega) = \widetilde{\Phi}(\eta,\omega;\lambda)(1+|\lambda|)^N e^{-R|\mathrm{Im}\lambda|}. \tag{6.4}$$

Then the above two estimates (6.2) and (6.3) show that $\{H_\lambda \mid \lambda \in \mathbb{C}\}$ is a bounded set in $C^\infty(F_{q,n})$.

Now we put $\widetilde{F}(\sigma,\omega;r) = \mathcal{F}_p f(\sigma, r\omega)$ as in (3.19). Then by the (projection-slice) Theorem 3.4,

$$\widetilde{\Phi}(\eta,\omega;r) = \int_{\sigma\subset\eta} \widetilde{F}(\sigma,\omega;r)\,d\sigma.$$

(This is the same as Eq. (3.11).) Just as with (3.18), for fixed ω the inversion formula (3.10) may again be applied to recover $\widetilde{F}$ from $\widetilde{\Phi}$:

$$\widetilde{F}(\sigma,\omega;r) = \square^{\omega}_{p,q}(S\widetilde{\Phi})(\sigma,\omega;r) = \square^{(p)} \circ S\widetilde{\Phi}(\sigma,\omega;r).$$

This formula still holds when r is replaced by a complex parameter λ.

$$\widetilde{F}(\sigma,\omega;\lambda) = \square^{(p)} \circ S\widetilde{\Phi}(\sigma,\omega;\lambda). \tag{6.5}$$

Next, we put $G_\lambda(\sigma,\omega) = \widetilde{F}(\sigma,\omega;\lambda)(1+|\lambda|)^N e^{-R|\mathrm{Im}\lambda|}$. Then $G_\lambda = \square^{(p)} \circ SH_\lambda$. Since by Lemmas 3.1 and 3.2 $\square^{(p)} \circ S$ is a continuous linear operator from $C^\infty(F_{q,n})$ to $C^\infty(F_{p,n})$, the set $\{G_\lambda \mid \lambda \in \mathbb{C}\}$ is bounded in $C^\infty(F_{p,n})$. In particular, we have

$$\sup_{(\sigma,\omega;\lambda)\in F_{p,n}\times\mathbb{C}} |G_\lambda(\sigma,\omega)| = \sup_{(\sigma,\omega;\lambda)\in F_{p,n}\times\mathbb{C}} |\widetilde{F}(\sigma,\omega;\lambda)(1+|\lambda|)^N e^{-R|\mathrm{Im}\lambda|}| < +\infty,$$

which shows that the function $\lambda \mapsto \widetilde{F}(\sigma,\omega;\lambda)$ satisfies the uniform Paley–Wiener estimate (6.1).

Next, we will prove that the function $\lambda \mapsto \widetilde{F}(\sigma,\omega;\lambda)$ is holomorphic in $\mathbb{C}$ for each $(\sigma,\omega) \in F_{p,n}$. Let us take any closed curve Γ in $\mathbb{C}$. Then by the continuity of $\square^{(p)} \circ S$,

$$\int_\Gamma F(\sigma,\omega;\lambda)\,d\lambda = \square^{(p)} \circ S\left(\int_\Gamma \widetilde{\Phi}(\cdot,\cdot;\lambda)\,d\lambda\right)(\sigma,\omega) = 0.$$

Here we used the fact that $\widetilde{\Phi}$ is holomorphic in λ. Therefore, by Morera's theorem, $\lambda \mapsto \widetilde{F}(\sigma, \omega; \lambda)$ is holomorphic.

To finish the proof of Theorem 6.2 it remains to prove that for each homogeneous spherical harmonic h_σ of degree k on the unit sphere S_σ in $\sigma^\perp$, the function

$$\lambda \;\mapsto\; \lambda^{-k} \int_{S_\sigma} \widetilde{F}(\sigma, \omega; \lambda)\, h_\sigma(\omega)\, d\omega \tag{6.6}$$

is even and holomorphic in λ. First, the fact that $\widetilde{F}(\sigma, \omega; \lambda) = \widetilde{F}(\sigma, -\omega, -\lambda))$ implies that (6.6) is even in λ.

In the calculations leading to Proposition 5.1, which use the forward moment condition (H'), the real parameter r can be replaced by the complex parameter λ to give us the following expression:

$$\widetilde{\Phi}(\eta, \omega; \lambda) = \sum_{l=0}^{k-1} \frac{(-i\lambda)^l}{l!} \int_{\sigma \subset \eta} P_l(\sigma, \omega)\, d\sigma + \int_{\eta^\perp} \varphi(\eta, y)(-i\lambda\langle y, \omega\rangle)^k g_k(-i\lambda\langle y, \omega\rangle)\, dy,$$

where $g_k(-iz) = e_k(-iz)/(-iz)^k$ is holomorphic in z and bounded if z is real. (See (5.1) for an explicit expression of g_k.) We write the last integral above as

$$\lambda^k \int_{\eta^\perp} \varphi(\eta, y)\, (-i\langle y, \omega\rangle)^k\, g_k(-i\lambda\langle y, \omega\rangle)\, dy \;=\; \lambda^k\, \Psi_k(\eta, \omega; \lambda),$$

where $\Psi_k(\eta, \omega; \lambda)$ is a smooth function on $F_{q,n} \times \mathbb{C}$ and holomorphic in λ for each $(\eta, \omega) \in F_{q,n}$.

By the inversion formula (6.5) (and (3.1) for the function $P_l \in C^\infty(G(p, n))$), we get

$$\widetilde{F}(\sigma, \omega; \lambda) \;=\; \sum_{l=0}^{k-1} \frac{(-i\lambda)^l}{l!}\, P_l(\sigma, \omega) \;+\; \lambda^k\, \Box^{(p)} S\Psi_k(\sigma, \omega; \lambda).$$

Similarly as in the proof of holomorphicity of $\widetilde{F}(\sigma, \omega; \lambda)$, we can prove that $\Box^{(p)} S\Psi_k (\sigma, \omega; \lambda)$ is holomorphic in λ for each (σ, ω). Hence

$$\begin{aligned}
\int_{S_\sigma} \widetilde{F}(\sigma, \omega; \lambda)\, h_\sigma(\omega)\, d\omega &= \sum_{l=0}^{k-1} \frac{(-i\lambda)^l}{l!} \int_{S_\sigma} P_l(\sigma, \omega)\, h_\sigma(\omega)\, d\omega \\
&\quad + \lambda^k \int_{S_\sigma} \Box^{(p)} S\Psi_k(\sigma, \omega; \lambda)\, h_\sigma(\omega)\, d\omega \\
&= \lambda^k \int_{S_\sigma} \Box^{(p)} S\Psi_k(\sigma, \omega; \lambda)\, h_\sigma(\omega)\, d\omega,
\end{aligned}$$

since $P_l(\sigma, \omega)$ is a degree l polynomial in $\omega \in S_\sigma$, and so is a sum of spherical harmonics in S_σ of degree $\leqslant l$. The fact that the mapping

$$\lambda \mapsto \lambda^{-k} \int_{S_\sigma} \widetilde{F}(\sigma, \omega; \lambda)\, h_\sigma(\omega)\, d\omega = \int_{S_\sigma} \Box^{(p)} S\Psi_k(\sigma, \omega; \lambda)\, h_\sigma(\omega)\, d\omega$$

is holomorphic in λ now follows. The function $\mathcal{F}_p f(\sigma, \lambda\omega) = \widetilde{F}(\sigma, \omega; \lambda)$ thus satisfies conditions (i) and (ii) in Theorem 6.1, and hence by that theorem f is supported in $\bar{\beta}_p^R$. This completes the proof of Theorem 6.2. $\Box$

7. Support Theorem 2

In this section, we prove the support theorem for the Radon transform $\mathcal{R}^{(p,q)} : \mathcal{S}(G(p,n)) \to \mathcal{S}(G(q,n))$ in the case when $p < q$ and $\dim G(p,n) < \dim G(q,n)$.

Let $\mathcal{O}$ be a subset in $\mathbb{R}^n$. Throughout this section, we assume the following condition (A) on $\mathcal{O}$.

$$\text{For any } \ell \in G(p,n) \text{ with } \ell \subset \mathcal{O}^c, \text{ there exists a hyperplane } L \text{ such that } \ell \subset L \subset \mathcal{O}^c. \tag{A}$$

The following support theorem holds:

Theorem 7.1. *Suppose that* $f \in \mathcal{S}(G(p,n))$. *If* $\mathcal{R}^{(p,q)} f(\xi) = 0$ *for all* $\xi \subset \mathcal{O}^c$, *then* $f(\ell) = 0$ *for all* $\ell \subset \mathcal{O}^c$.

As a corollary, we have the usual support theorem for $\mathcal{R}^{(p,q)}$, namely,

Corollary 7.1. *Let* $r > 0$. *Suppose that* $f \in \mathcal{S}(G(p,n))$. *If* $\mathcal{R}^{(p,q)} f(\xi) = 0$ *for all* ξ, $\operatorname{dist}(\xi, 0) > r$, *then* $f(\ell) = 0$ *for all* ℓ, $\operatorname{dist}(\ell, 0) > r$.

Proof. If $\mathcal{O} = \{\, x \in \mathbb{R}^n \mid ||x|| < r \,\}$, then obviously $\mathcal{O}$ satisfies condition (A). $\Box$

Remark 7.1. Similarly, if $\mathcal{O}$ is a convex set, then $\mathcal{O}$ satisfies condition (A). So, in this case, the support theorem also holds.

However, there are many cases when $\mathcal{O}$ satisfies condition (A) but $\mathcal{O}$ is not necessarily convex. Even in such a case, the support theorem holds.

The following are examples.

Example 1. For a, b $(a < b)$ and for $r > 0$, let

$$\begin{aligned} \mathcal{O}_1 &= \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid (x_1 - a)^2 + x_2^2 + \cdots + x_n^2 \leqslant r^2,\ x_1 \leqslant a \,\}, \\ \mathcal{O}_2 &= \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid (x_1 - b)^2 + x_2^2 + \cdots + x_n^2 \leqslant r^2,\ x_1 \geqslant b \,\} \end{aligned} \tag{7.1}$$

and let $\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2$. Then, $\mathcal{O}$ is no longer convex. But it is easily seen that $\mathcal{O}$ satisfies the condition (A).

Example 2. For a, b $(a < b)$, let

$$\begin{aligned} \mathcal{O}_1 &= \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_1 \leqslant a \}, \\ \mathcal{O}_2 &= \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_1 \geqslant b \}, \end{aligned} \tag{7.2}$$

and let $\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2$. Then, $\mathcal{O}^c$ is a band domain. Similarly as in the above example, $\mathcal{O}$ is not convex, but $\mathcal{O}$ satisfies condition (A).

Now, we proceed to prove Theorem 7.1. The key is the injectivity of the Radon transform.

Proof of Theorem 7.1. Let us take an arbitrary p-plane ℓ_0 which is included in $\mathcal{O}^c$. Then, by condition (A), there exists a hyperplane $L \in G(n-1, n)$ such that $\ell_0 \subset L \subset \mathcal{O}^c$. Let $G(d, n; L) = \{ \gamma \in G(d, n) \mid \gamma \subset L \}$. Then the space of Schwartz class functions $\mathcal{S}(G(d, n; L))$ on $G(d, n; L)$ is defined in a similar manner to $\mathcal{S}(G(d, n))$. In addition, we can define a Radon transform $\mathcal{R}_L^{(p,q)} : \mathcal{S}(G(p, n; L)) \to \mathcal{S}(G(q, n; L))$ as follows:

$$\mathcal{R}_L^{(p,q)} f(\xi) = \int_{\ell \subset \xi} f(\ell)\, d_\xi \ell \quad \text{for } \xi \in G(q, n; L) \quad \text{and for } f \in \mathcal{S}(G(q, n; L)), \tag{7.3}$$

where $d_\xi \ell$ is the canonical measure on the set $\{\ell \in G(p, n; L) \mid \ell \subset \xi \}$. For $f \in \mathcal{S}(G(p, n))$ in the statement of Theorem 7.1, let f_L be the restriction of f onto the submanifold $G(p, n; L)$. Then, $f_L \in \mathcal{S}(G(p, n; L))$. Moreover, by the definition of $\mathcal{R}_L^{(p,q)}$ and by the assumption of the theorem, we have

$$\mathcal{R}_L^{(p,q)} f_L(\xi) = \mathcal{R}^{(p,q)} f(\xi) = 0 \quad \text{for } \xi \in G(q, n; L). \tag{7.4}$$

In fact, if $\xi \in G(q, n; L)$, then $\xi \subset \mathcal{O}^c$ and therefore $\mathcal{R}^{(p,q)} f(\xi) = 0$.

By applying a suitable translation and a suitable orthogonal transformation to L, we may assume that $L = \mathbb{R}\mathbf{e}_1 \oplus \cdots \oplus \mathbb{R}\mathbf{e}_{n-1} (\cong \mathbb{R}^{n-1})$. Then, the Radon transform $\mathcal{R}_L^{(p,q)}$ is nothing but the Radon transform from $\mathcal{S}(G(p, n-1))$ to $\mathcal{S}(G(q, n-1))$ associated with inclusion incidence relation. By the assumption that $p < q$ and $\dim G(p, n) < \dim G(q, n)$, we see easily that $\dim G(p, n-1) \leqslant \dim G(q, n-1)$. So, by Theorem 6.4 and Remark 3 in Section 6 of our previous paper [GK1], $\mathcal{R}_L^{(p,q)}$ is injective. Hence, by (7.4), $f_L(\ell) = 0$ for $\ell \in G(p, n; L)$. In particular, $f(\ell_0) = f_L(\ell_0) = 0$, which completes the proof. $\square$

Finally, as an application of the above support theorem, we give a range characterization of $\mathcal{R}^{(p,q)}$ in the category of compactly supported smooth functions.

Let $C_c^\infty(G(d,n))$ denote the space of compactly supported smooth functions on $G(d,n)$. Then, it is easily seen that since $p < q$ the Radon transform $\mathcal{R}^{(p,q)}$ maps $C_c^\infty(G(p,n))$ to $C_c^\infty(G(q,n))$. As we stated in the introduction, since $\dim G(p,n) < \dim G(q,n)$ the image of $\mathcal{R}^{(p,q)}$ is characterized as the solution space of an $M(n)$-invariant differential equation of order $2p+4$ of the form,

$$dv(Q_{2p+4})\,\varphi \;=\; 0. \tag{7.5}$$

Here Q_{2p+4} is an element in $\mathfrak{z}(\mathfrak{m}(n))$ and is expressed as the sum of the squares of Pfaffians" of order $p+2$. (See Gonzalez and Kakehi [GK1] for the definition of the operator Q_{2p+4} and its detailed properties.)

Since $C_c^\infty(G(d,n)) \subset \mathcal{S}(G(p,n))$, the support theorem (Theorem 7.1) and the range theorem for $\mathcal{R}^{(p,q)}$ [GK1, Theorem 7.7] yield the following:

Theorem 7.2. *Suppose that* $p < q$ *and* $\dim G(p,n) < \dim G(q,n)$. *A function* $\varphi \in C_c^\infty(G(q,n))$ *belongs to the range* $\mathcal{R}^{(p,q)}(C_c^\infty(G(p,n)))$ *if and only if* $dv(Q_{2p+4})\,\varphi \;=\; 0$.

References

[GGV] I.M. Gelfand, M.I. Graev, N. Vilenkin, Generalized Functions, vol. 5, Integral Geometry and Representation Theory, Academic Press, New York, 1966.

[Go1] F. Gonzalez, Bi-invariant differential operators on the Euclidean motion group and applications to generalized Radon transforms, Ark. Mat. 26 (1988) 191–204.

[Go2] F. Gonzalez, Invariant differential operators and the range of the Radon d-plane transform, Math. Ann. 287 (1990) 627–635.

[Go3] F. Gonzalez, On the range of the Radon d-plane transform and its dual, Trans. Amer. Math. Soc. 327 (1991) 601–619.

[GH] F. Gonzalez, S. Helgason, Invariant differential operators on Grassmann manifolds, Adv. Math. 60 (1986) 81–91.

[GK1] F. Gonzalez, T. Kakehi, Pfaffian systems and Radon transforms on affine Grassmann manifolds, Math. Ann. 326 (2003) 237–273.

[GK2] F. Gonzalez, T. Kakehi, Dual Radon transforms on affine Grassmann manifolds, Trans. Amer. Math. Soc. 356 (2004) 4161–4180.

[Gr] E. Grinberg, Radon transforms on higher rank Grassmannians, J. Differential Geom. 24 (1986) 53–68.

[GrRu] E. Grinberg, B. Rubin, Radon inversion on Grassmannians via Garding–Gindikin fractional integrals, Ann. Math. 158 (2004) 783–817.

[H1] S. Helgason, The Radon transform on Euclidean spaces, two-point homogeneous spaces, and Grassmann manifolds, Acta Math. 113 (1965) 153–180.

[H2] S. Helgason, A duality for symmetric spaces with applications to group representations, II. Differential equations and eigenspace representations, Adv. Math. 22 (1976) 187–219.

[H3] S. Helgason, Groups and Geometric Analysis, Academic Press, Orlando, 1984.

[H4] S. Helgason, Geometric Analysis on Symmetric Spaces, American Mathematical Society, Mathematical Surveys and Monographs, vol. 39, Providence, 1994.

[H5] S. Helgason, The Radon Transform, second ed., Progress in Mathematics, vol. 5, Birkhäuser, Boston, 1999.

[J] F. John, The ultrahyperbolic differential equation with 4 independent variables, Duke Math. J. 4 (1938) 300–322.

[K] T. Kakehi, Integral geometry on Grassmann manifolds and calculus of invariant differential operators, J. Funct. Anal. 168 (1999) 1–45.

[KN] S. Kobayashi, K. Nomizu, Foundations of Differential Geometry I, II, Wiley, New York, 1963, 1969.

[Lu] D. Ludwig, The Radon transform on Euclidean space, Comm. Pure Appl. Math. 23 (1966) 49–81.

[Ri] F. Richter, On the k-dimensional Radon transform of rapidly decreasing functions, Lecture Notes in Mathematics, vol. 1209, Springer, Berlin, New York, 1986.

[Ru1] B. Rubin, Inversion and characterization of the hemispherical transform, J. D'Anal. Math. 77 (1999) 105–128.

[Ru2] B. Rubin, Radon transforms on affine Grassmannians, Trans. Amer. Math. Soc. 356 (2004) 5045–5070.

[St] R. Strichartz, Harmonic analysis on Grassmann bundles, Trans. Amer. Math. Soc. 296 (1986) 387–409.