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The $\mathscr{C}$-Spectral Sequence, Lagrangian Formalism, and Conservation Laws. II. The Nonlinear Theory

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Contents. Introduction. 6. Differential calculus on infinitely prolongated equations. 7. Spencer complexes in $\mathscr{C}$-theory. 8. Nonlinear Lagrangian formalism. 9. The $\mathscr{C}$-spectral sequence. 10. The $\mathscr{C}$-spectral sequence of infinitely prolongated equations. 11. Applications to the Lagrangian formalism with constraints and conservation laws theory. 12. Concluding remarks.

INTRODUCTION

This is the second part of the article which immediately follows Part I in this issue. Here the main "nonlinear" constructions are introduced and the main results obtained. The general introduction to both parts can be found in Part I. Within the paper a unique system of notations, of numeration of sections, etc., is adopted. Accordingly, a common system of references is used in both parts.

Briefly this part is organized as follows: General facts concerning infinite jet manifolds and infinitely prolonged differential equations are put together in Section 6. These are used to develop a specific differential calculus on such manifolds. In particular, an important class of differential operators, the so-called $\mathscr{C}$-differential operators, is described here. In Section 7 we generalize the Spencer complexes and related machinery to $\mathscr{C}$-differential operators. General concepts of nonlinear Lagrangian formalism are introduced in Section 8, together with main formulas. These are used to obtain a very general version of the Noëther theorem. Here we also give an exposition of the general theory of "transversality conditions."

The main object of this paper, the $\mathscr{C}$-spectral sequence $\{E_r^{p,q}, d_r^{p,q}\}$, is defined in Section 9 and its general properties are derived. Then, using the generalised Spencer complexes, we calculate the E_1-term of the $\mathscr{C}$-spectral sequence and show its relation to some of the "fundamental problems" of the calculus of variations. In particular, we solve the global inverse problem of the calculus of variations as well as the triviality problem for Lagrangians.

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The structure of the $\mathscr{C}$-spectral sequence of an infinitely prolongated equation is investigated in Section 10. Using Spencer cohomology-type techniques we obtain here a description of its E_1-term. This is the heart of the paper. Finally, these structural results are applied in Section 11 to conservation laws and the Lagrangian formalism with constraints. Here we collect some formulas, useful in practical calculations. In the last paragraph we briefly discuss some possible generalisations and other applications of the $\mathscr{C}$-spectral sequences.

6. The Differential Calculus on Infinitely Prolongated Equations

In this section we bring together the necessary information from the theory of jet spaces, on which our further exposition is based. Details may be found in [4, 13].

6.1. *Jet Manifolds*

Suppose N is a smooth manifold, $\dim N = m + n$, $m \geqslant 1$. The class of n-dimensional submanifolds $L \subset N$ tangent to each other at the point x with order k will be called a k-jet of the n-dimensional submanifold of N; the k-jet of the submanifold L at the point $x \in L$ will be denoted by $[L]_x^k$. Suppose $N_m^k(x) = \{[L]_x^k \mid \dim L = n, x \in L\}$ and $N_m^k = \bigcup_{x \in N} N_m^k(x)$. The set N_m^k has a natural smooth manifold structure. For $k \geqslant l$ we have the projections

$$\pi_{k,l}\colon N_m^k \to N_m^l, \qquad \pi_{k,l}([L]_x^k) = [L]_x^l.$$

If $L^n \subset N$ then the smooth map

$$j_k(L)\colon L \to N_m^k, \qquad j_k(L)(x) = [L]_x^k$$

is defined and we have $\pi_{k,l} \circ j_k(L) = j_l(L)$. The inverse limit of the chain of maps

$$\cdots \to N_m^k \xrightarrow{\pi_{k,k-1}} N_m^{k-1} \to \cdots \xrightarrow{\pi_{1,0}} N_m^0 = N$$

will be denote by N_m^∞, the limit of the system of maps $j_k(L)$, $k \to \infty$, by $j(L)$ and the natural projection $N_m^\infty \to N_m^k$ by $\pi_{\infty,k}$. The set N_m^k is called the manifold of k-jets of n-dimensional submanifolds in N, $0 \leqslant k \leqslant \infty$ and $j_k(L)$ (or $j(L)$) is the graph of the k-jet (∞-jet) of the submanifold L. We also put $\mathscr{F}_m^k(N) = C_{\mathbb{F}}^\infty(N_m^k)$ and denote by $\mathscr{F}_m(N)$ the direct limit of rings

$$C_{\mathbb{F}}^\infty(N) = \mathscr{F}_m^0(N) \xrightarrow{\pi_{1,0}^*} \cdots \xrightarrow{\pi_{k-1,k-2}^*} \mathscr{F}_m^{k-1}(N) \xrightarrow{\pi_{k,k-1}^*} \mathscr{F}_m^k(N) \to \cdots.$$

Suppose

$$\pi\colon E_\pi \to M^n, \qquad \dim E_\pi = m + n,$$

is a submersion. The set of k-jets of local sections of the submersion π constitutes an open set in $(E_\pi)^k_m$ which will be denoted by $J^k(\pi)$. In the case when π is a fiber bundle, $J^k(\pi)$ is said to be the space of k-jets of this bundle. If $\sigma \in \Gamma_{\text{loc}}(\pi)$ and $\mathscr{U}$ is the domain of σ we define $j_k(\sigma) = j_k(L) \circ \sigma\colon \mathscr{U} \to J^k(\pi)$, where $L = \sigma(\mathscr{U})$. We also put

$$\pi_k = \pi \circ \pi_{k,0}\colon J^k(\pi) \to M, \quad k \leqslant \infty; \qquad \mathscr{F}_k(\pi) = C^\infty_{\mathbb{F}}(J^k(\pi)),$$
$$\mathscr{F}_{-1}(\pi) = C^\infty(M), \qquad \mathscr{F}(\pi) = \lim\operatorname{dir} \mathscr{F}_k(\pi).$$

Consider the manifold $\mathscr{U} = V^n \times W^m$, where V^n (resp. W^n) is a domain in $\mathbb{R}^n$ (resp. $\mathbb{R}^m$) and suppose (x, u),

$$x = (x_1, ..., x_n) \in \mathbb{R}^n, \qquad u = (u_1, ..., u_m) \in \mathbb{R}^m$$

is the corresponding coordinate system. Then on the manifold $J^k(\alpha)$, $0 \leqslant k \leqslant \infty$, $\alpha\colon \mathscr{U} \to V$ is the natural projection, a special coordinate system

$$x_j, p^i_\tau, \qquad 1 \leqslant j \leqslant n,\ 1 \leqslant i \leqslant m,\ |\tau| = k$$

arises, where τ is the non-ordered sequence of integers $(i_1, ..., i_r)$, $1 \leqslant i_a \leqslant n$, $|\tau| = r$. The functions p^i_τ are entirely determined by their property

$$j_k(s)^*\,(p^i_\tau) = \frac{\partial^{|\tau|} s_i}{\partial x_{i_1} \cdots \partial x_{i_r}},$$

where the section $s \in \Gamma_{\text{loc}}(\alpha)$ is given by the relation $u_i = s_i(x)$. In particular, $u_i = p^i_\varnothing$.

Suppose $f\colon \mathscr{U} \to \mathscr{U}' \subset N$ (or E_π) is a diffeomorphism. Then the diffeomorphisms $f_{(k)}\colon J^k_{(\alpha)} \to N^k_m$ are naturally defined; using them we can carry over the special coordinates from $J^k(\alpha)$ to $\operatorname{im} f_{(k)}$.

The family of maps $\{f_{(k)}\}$, $0 \leqslant k \leqslant \infty$, and sometimes its image will be called a chart on N^∞_m (or $J^\infty(\pi)$). If the map $f\colon \mathscr{U} \to E_\pi$ is also a morphism of the bundle $V \times W \to V$ into the bundle π we shall say that the chart $\{f_{(k)}\}$ is compatible with π.

6.2. *Infinitely Prolongated Equations*

The system of nonlinear differential equations (or simply "equation"), imposed on the n-dimensional submanifolds of the manifold N (the sections of the bundle π) may be understood as a submanifold $\mathscr{Y} \subset N^k_m (\mathscr{Y} \subset J^k(\pi))$. Denote by $\mathscr{Y}^s \subset N^{k+s}_m$ $(\mathscr{Y}^s \subset J^{k+s}(\pi))$ the sth prolongation of $\mathscr{Y}$ (see, e.g.,

[33, 34]). Then $\pi_{s+k,t+k}$ maps $\mathscr{Y}^s$ into $\mathscr{Y}^t$, $s \geqslant t$. The inverse limit of the chain of maps $\mathscr{Y} = \mathscr{Y}^0 \xleftarrow{\pi_{k+1,k}} \mathscr{Y}^1 \xleftarrow{\pi_{k+2,k+1}} \cdots$ will be denoted by $\mathscr{Y}_\infty$. Obviously, $\mathscr{Y}_\infty \subset N_m^\infty$ ($\mathscr{Y}_\infty \subset J^\infty(\pi)$) and $\mathscr{Y}_s = \pi_{\infty,s}(\mathscr{Y}_\infty) \subset \mathscr{Y}^{s-k}$. Suppose $\mathscr{F}_s(\mathscr{Y}) = C^\infty_{\mathbb{F}}(N_m^s)|_{\mathscr{Y}_s}$ (or $= C^\infty_{\mathbb{F}}(J^s(\pi)|_{\mathscr{Y}_s})$ and let $\mathscr{F}(\mathscr{Y})$ be the direct limit of the chain of maps $\cdots \mathscr{F}_s(\mathscr{Y}) \xrightarrow{i^*_{s,t}} \mathscr{F}_{s+1}(\mathscr{Y}) \to \cdots$, where $i_{s,t} = \pi_{s,t}|_{\mathscr{Y}_s}$: $\mathscr{Y}_s \to \mathscr{Y}_t$ is surjective. Therefore we may assume that the $\mathbb{F}$-algebra $\mathscr{F}(\mathscr{Y})$ is filtered by the subalgebras $\mathscr{F}_s(\mathscr{Y})$. Note that for formally integrable equations we have $\mathscr{Y}_s = \mathscr{Y}^{s-k}$.

Suppose $\eta: E_\eta \to N$ is a smooth bundle $\eta_s = \pi^*_{s,0}(\eta)$, $\eta_s(\mathscr{Y}) = \eta_s|_{\mathscr{Y}_s}$ and $\mathscr{F}_s(\mathscr{Y}, \eta) = \Gamma(\eta_s(\mathscr{Y}))$. The homomorphisms $\pi^*_{s,t}$ generate monomorphisms $i^*_{s,t} = i_{s,t}(\mathscr{Y}, \eta)^*: \mathscr{F}_t(\mathscr{Y}, \eta) \to \mathscr{F}_s(\mathscr{Y}, \eta)$ whose direct limit will be denoted by $\mathscr{F}(\mathscr{Y}, \eta)$. If η is a vector bundle, then $\mathscr{F}(\mathscr{Y}, \eta)$ is a $\mathscr{F}(\mathscr{Y})$-module filtered by the $\mathscr{F}_s(\mathscr{Y})$-modules $\mathscr{F}_s(\mathscr{Y}, \eta)$. Suppose $\mathscr{Y} = N_m^k$. Then $\mathscr{Y}_s = N_m^s$ and we shall use the notation $\mathscr{F}_s(N_m^\infty, \eta) = \mathscr{F}_s(\mathscr{Y}, \eta)$, $\mathscr{F}(N_m^\infty, \eta) = \mathscr{F}(\mathscr{Y}, \eta)$. In the particular case when $\mathscr{Y} \subset J^k(\pi)$ and $\xi: E_\xi \to M$ is a smooth bundle we shall write $\mathscr{F}_s(\mathscr{Y}, \xi)$ instead of $\mathscr{F}_s(\mathscr{Y}, \pi^*(\xi))$ and also put

$$\mathscr{F}_{-1}(\mathscr{Y}, \xi) = \Gamma(\xi), \qquad \mathscr{F}_{-1}(\mathscr{Y}) = C^\infty(M),$$
$$\mathscr{F}_s(\pi, \xi) = \Gamma(\pi_s^*(\xi)), \qquad \mathscr{F}(\pi, \xi) = \lim_{s\to\infty} \operatorname{dir} \mathscr{F}_s(\pi, \xi).$$

Any section $\varphi \in \mathscr{F}_s(\pi, \xi)$ generates a non-linear differential operator of order $\leqslant s$ $\Delta = \Delta_\varphi: \Gamma(\pi) \to \Gamma(\xi)$ where $\Delta(\sigma) = j_s(\sigma)^*(\varphi)$. This correspondence is bijective. The section from $\mathscr{F}_s(\pi, \xi)$ which corresponds to the operator $\Delta: \Gamma(\pi) \to \Gamma(\xi)$ under this bijection will be denoted by φ_Δ.

6.3. *The FG-Category*

Suppose A is a commutative K-algebra, filtered by the subalgebras $\cdots \subset A_i \subset A_{i+1} \subset \cdots$, $A = \bigcup A_i$ P and Q are A-modules, filtered by the A_i-submodules P_i and Q_i, respectively, and $P = \bigcup P_i$, $Q = \bigcup Q_i$. The operator $\Delta \in \operatorname{Diff}(P, Q)$ is said to be filtered if there exists an integer s such that $\Delta(P_i) \subset Q_{i+s}$, $\forall i$. The filtered A-module P will be called geometrical if for any i the A_i-module P_i is geometrical. The category whose objects are filtered geometric A-modules P, $P = \bigcup P_i$ and the morphisms are filtered differential operators (d.o.) will be called the *FG*-category. It is differentially closed. The representative objects for the functors of the differential calculus in this category exist and are direct limits of the corresponding objects for the theory of geometric d.o. over the algebra A_i. Further for A we shall take the $\mathbb{F}$-algebras $\mathscr{F}_m(N)$, $\mathscr{F}(\pi)$, or $\mathscr{F}(\mathscr{Y})$. Typical A-modules in this case were described in the previous subsection.

Further the set of morphisms in the *FG*-category are simply denoted by $\operatorname{Diff}(P, Q)$ and the usual notations are preserved for the representative

objects. In particular, Λ^* denotes the A-module of differential forms in this category.

6.4. *The Operation* $\mathscr{C}$

Suppose Φ is some functor of the differential calculus over $\mathscr{F}(\mathscr{Y})$, $\mathscr{F}_m(N)$, or $\mathscr{F}(\pi)$ while $\varPhi$ is its representative object in the FG-category. Put

$$\mathscr{C}\varPhi = \{\varphi \in \varPhi \mid [j(L)^*(\varphi)](x) = 0, \text{ if } j(L)(x) \in \mathscr{Y}_\infty\}.$$

If L is a solution of the equation $\mathscr{Y}$, i.e., $\operatorname{im} j_k(L) \subset \mathscr{Y}$, then $j(L)(\mathscr{C}\varPhi) = 0$. The natural differential operators are compatible with the operation $\mathscr{C}$. In particular, $d(\mathscr{C}\Lambda^i) \subset \mathscr{C}\Lambda^{i+1}$. For this reason the quotients $\bar{\varPhi} = \varPhi/\mathscr{C}\varPhi$ are supplied with quotient operators, which will be denoted by adding an overbar denoting the original operator. For example, $\bar{d}: \bar{\Lambda}^i \to \bar{\Lambda}^{i+1}$. Obviously, $\{\bar{\Lambda}^*, \bar{d}\}$ is a complex. Its cohomology will further be denoted by $\bar{H}^i$. For example, $\bar{H}^i(\mathscr{Y}_\infty)$ or simply $\bar{H}^i(\mathscr{Y})$.

The operation $\mathscr{C}$ allows us to consider, together with every functor Φ of the differential calculus, the subfunctor $\mathscr{C}\Phi = \operatorname{Ann} \mathscr{C}\varPhi$. To be more exact,

$$\mathscr{C}\Phi(P) = \{h \in \Phi(P) = \operatorname{Hom}(\varPhi, P) \mid h(\mathscr{C}\varPhi) = 0\}.$$

The representative object for the functor $\mathscr{C}\Phi$ is therefore $\bar{\varPhi}$. The theory which arises in this way will further be called $\mathscr{C}$-theory. In particular if $\Phi: Q \mapsto \operatorname{Diff}_s(P, Q)$ (the module P is fixed) then the operator $\Delta \in \mathscr{C} \operatorname{Diff}_s(P, Q)$ will be called $\mathscr{C}$-differential.

The notion of $\mathscr{C}$-differential operator is intimately related to the notion of prolongation of an equation $\mathscr{Y} \subset N_m^k$ (or $J^k(\pi)$). Namely, suppose

$$\mathscr{I} \subset \mathscr{F}_m^k(N) \subset \mathscr{F}_m(N) \qquad \text{or} \qquad \mathscr{I} \subset \mathscr{F}_k(\pi) \subset \mathscr{F}(\pi)$$

is the ideal of the submanifold $\mathscr{Y}$. Then the ideal determined by $\mathscr{Y}_\infty$ in N_m^∞ (or $J^\infty(\pi)$) is the $\mathscr{C}$-radical closure of the ideal $(\mathscr{C} \operatorname{Diff} A)(\mathscr{I})$, $A = \mathscr{F}_m(N)$ or $= \mathscr{F}(\pi)$. This means that the ideal coincides with its own radical and is stable with respect to the action of operators from $\mathscr{C} \operatorname{Diff} A$.

The class of $\mathscr{C}$-differential operators coincides with the class of those differential operators in the FG-category which are compatible with all possible maps of the form $j(L)^*$ (or $j(s)^*$, $s \in \Gamma_{\mathrm{loc}}(\pi)$). This means that for every $\Delta \in \mathscr{C} \operatorname{Diff}(P, Q)$ and $L \subset N$ $(s \in \Gamma_{\mathrm{loc}}(\pi))$ the operator $\Delta|_L \in \operatorname{Diff}(j(L)^*(P), j(L)^*(Q))$ over $C^\infty(L)$ (resp. $\Delta|_s \in \operatorname{Diff}(j(s)^*(P), j(s)^*(Q))$ over $C^\infty(\mathscr{U})$, where $\mathscr{U}$ is the domain of s) is defined and satisfies

$$\Delta|_L \circ j(L)^* = j(L)^* \circ \Delta \qquad \text{or} \qquad \Delta|_s \circ j(s)^* = j(s)^* \circ \Delta.$$

This fact immediately follows from the definitions and has the following consequence: every operator of the form $\bar{\Delta}: \bar{\varPhi}_1 \to \bar{\varPhi}_2$, where $\Delta: \varPhi_1 \to \varPhi_2$, is

some natural operator on the representative objects is $\mathscr{C}$-differential. For example, the operator $\bar{d}\colon \bar{\Lambda}^i \to \bar{\Lambda}^{i+1}$ is $\mathscr{C}$-differential.

It follows from the property of $\mathscr{C}$-differential operators noted above that every $\Delta \in \mathscr{C}\ \mathrm{Diff}(P, Q)$ possesses a restriction to subobjects from DE.

Our approach to the nonlinear Lagrange formalism is natural in the category DE of nonlinear differential equations whose informal description is the following. The objects of DE are "manifolds" of the form $\mathscr{Y}_\infty$, or to be more precise, filtered algebras of the form $\mathscr{F}(\mathscr{Y})$ in which the differential calculus is provided with the operation $\mathscr{C}$, and also their sub- and quotient algebras which inherit this operation. The morphisms in DE are homomorphisms of filtered algebras of the type indicated which preserve the operation $\mathscr{C}$.

6.5. *The Operation* $\urcorner$

Let us assume that $\mathscr{Y} \subset J^k(\pi)$ and therefore $\mathscr{Y}_\infty \subset J^\infty(\pi)$. The $\mathscr{F}(\mathscr{Y})$-module of the form $\mathscr{F}(\mathscr{Y}) \otimes_{C^\infty(M)} \Gamma(\pi')$, where $\pi'\colon E_{\pi'} \to M$ is a vector bundle will be called horizontal. If Φ represents the functor Φ in the *FG*-category over $\mathscr{F}(\mathscr{Y})$, while $\Phi(M)$ represents it in the geometric category over $C^\infty_{\mathbb{F}}(M)$ we put

$$\Phi_0 = \mathscr{F}(\mathscr{Y}) \otimes_{C^\infty(M)} \Phi(M).$$

Further the elements $\varphi \in \Phi_0 \subset \Phi$ will be called horizontal. For any Φ there also exists a unique operation of horizontalization $\urcorner\colon \Phi \to \Phi_0$ defined by the equality

$$[j(s)^* (\urcorner\varphi)](x) = [j(s)^* (\varphi)](x), \qquad s \in \Gamma_{\mathrm{loc}}(\pi),\ j(s)(x) \in \mathscr{Y}_\infty.$$

Obviously $\urcorner^2 = \urcorner$. Moreover, this operation splits the exact sequence of $\mathscr{F}(\mathscr{Y})$-modules $0 \to \mathscr{C}\Phi \leftrightarrows^{\mathrm{id}-\urcorner} \Phi \to \bar{\Phi} \to 0$ so that $\bar{\Phi}$ in this case can be naturally identified with $\ker(\mathrm{id} - \urcorner) = \mathrm{im}\,\urcorner = \Phi_0$. In the case $\Phi = \Lambda^*$ in special coordinates, compatible with the projection π the operation $\urcorner$ is entirely determined by the formulas

$$\urcorner(dp^i_\tau) = \sum_k p^i_{\tau k}\, dx_k, \qquad \urcorner(\omega_1 \wedge \omega_2) = \urcorner(\omega_1) \wedge \urcorner(\omega_2).$$

To every operator $\Delta \in \mathrm{Diff}_s(P, Q)$ we can assign the operator $\ulcorner\!\!\urcorner{\Delta} \in \mathscr{C}\ \mathrm{Diff}_s(P, Q)$ by putting $\ulcorner\!\!\urcorner{\Delta}(p) = (\Delta, \urcorner j_s(p))$, where the brackets $(\cdot, \cdot)$ denote the natural pairing of $\mathscr{J}^s(P)$ and $\mathrm{Diff}_s(P, Q)$. Then we have

$$\mathscr{C}\ \mathrm{Diff}_s(P, Q) = \{\ulcorner\!\!\urcorner{\Delta} \mid \Delta \in \mathrm{Diff}_s(P, Q)\}$$

and the operator $\ulcorner\!\!\urcorner{\Delta}$ is uniquely determined by its restriction to P_{-1}.

Consider the operator $\Sigma_{0,s}\colon \mathcal{J}^s(P) \to \mathcal{J}^{s-1}(P) \otimes \Lambda^1$ from the jet-Spencer sequence in the FG-category over $\mathscr{F}(\mathscr{Y})$ (see 2.3) and for every $p \in P$ put

$$U_s(p) = \Sigma_{0,s}(\square j_s(p)) \in \mathscr{C}\mathcal{J}^{s-1}(P) \otimes \mathscr{C}\Lambda^1.$$

In particular, if $P = \mathscr{F}(\mathscr{Y})$ and $s = 1$, we have $U_1(p) \in \mathscr{C}\Lambda^1$.

The forms $U_1(\varphi)$, $\varphi \in \mathscr{F}(\mathscr{Y})$ generate the $\mathscr{F}(\mathscr{Y})$-module $\mathscr{C}\Lambda^1$.

6.6. *Infinitesimal Symmetries*

Suppose A is one of the algebras $\mathscr{F}(\mathscr{Y})$, $\mathscr{F}(\pi)$, $\mathscr{F}_m(N)$. A derivation $X \in D(A)$ will be called an infinitesimal symmetry of the equation $\mathscr{Y}$ if $X(\mathscr{C}\Phi) \subset \mathscr{C}\Phi$ for all representative objects Φ. If $A = \mathscr{F}(\pi), \mathscr{F}_m(N)$ then X is said to be $\mathscr{C}$-field. Denote the set of all such symmetries ($\mathscr{C}$-fields) by $D_{\mathscr{C}}(A)$. Then $\mathscr{C}D(A)$ is an ideal in the Lie algebra $D_{\mathscr{C}}(A)$. The quotient algebra $D_{\mathscr{C}}(A)/\mathscr{C}D(A)$ will be denoted by $\mathrm{Sym}\,\mathscr{Y}$, $\kappa = \kappa(\pi)$, or $\kappa = \kappa(N_m^\infty)$, respectively, for $A = \mathscr{F}(\mathscr{Y})$, $\mathscr{F}(\pi)$, or $\mathscr{F}_m(N)$; note that $\kappa(N_m^\infty)$ or $\kappa(\pi)$ are also A-modules.

Note that the fact that $\mathscr{C}\,\mathrm{Diff}\,A$ is generated by $\mathscr{C}D(A)$ and A implies that the normalizer $N(A)$ of the Lie subalgebra $\mathscr{C}\,\mathrm{Diff}\,A$ in $\mathrm{Diff}\,A$ contains $D_{\mathscr{C}}(A)$, i.e., $[X, \Delta] \in \mathscr{C}\,\mathrm{Diff}\,A$ whenever $\Delta \in \mathscr{C}\,\mathrm{Diff}\,A$, $X \in D_{\mathscr{C}}(A)$. It can be shown that the natural map $D_{\mathscr{C}}(A)/\mathscr{C}D(A) \to N(A)/\mathscr{C}\,\mathrm{Diff}\,A$ is an isomorphism (see [13]).

Suppose $\chi \in \mathrm{Sym}\,\mathscr{Y}$ (or $\kappa(\pi)$), where $\mathscr{Y} \subset J^k(\pi)$, $\chi = X \mod \mathscr{C}D(A)$, $X \in D_{\mathscr{C}}(A)$. If we understand the element $U_s(p)$ as a jet-valued 1-form we can define the substitution operation by setting $\chi \lrcorner U_s(p) = X \lrcorner U_s(p)$. This operation is well defined, since $U_s(p) \in \mathscr{C}\mathcal{J}^{s-1}(P) \otimes \mathscr{C}\Lambda^1$. Let us now introduce the universal linearization operator $l_p\colon \kappa(\pi) \to P$ corresponding to $p \in P$ in the case when $A = \mathscr{F}(\pi)$ by putting

$$l_p(\chi) = \chi \lrcorner U_1(p), \qquad \chi \in \kappa(\pi),\ p \in P. \tag{6.6.1}$$

It turns out that $l_p \in \mathscr{C}\,\mathrm{Diff}(\kappa, P)$ and

$$l_{fp}(\chi) = f l_p(\chi) + l_f(\chi)p, \qquad f \in A.$$

It follows from the last relation that the operator $\mathfrak{Z}_\chi\colon P \to P$, $\mathfrak{Z}_\chi(p) = l_p(\chi)$ known as the operator of evolutionary derivation satisfies $\mathfrak{Z}_\chi(fp) = f\mathfrak{Z}_\chi(p) + \mathfrak{Z}_\chi(f)p$. Note that for $P = A$ we have $\mathfrak{Z}_\chi \in D_{\mathscr{C}}(A)$ and for every $X \in D_{\mathscr{C}}(A)$ we have the decomposition $X = \mathfrak{Z}_\chi + (X - \mathfrak{Z}_\chi)$, where $\chi = X \mod \mathscr{C}D(A)$ which implies the direct sum decomposition of the A-module $D_{\mathscr{C}}(A)$ to the form $\kappa \oplus \mathscr{C}D(A)$.

If the bundle π is linear, then $\kappa(\pi)$ can be identified naturally with $\mathscr{F}(\pi, \pi)$ by means of the map $\chi \mapsto \chi \lrcorner U_1(\pi)$, where $U_1(\pi) = U_1(\varphi_{\mathrm{id}})$, $\mathrm{id}\colon \Gamma(\pi) \to \Gamma(\pi)$ is the identity operator (see 6.2). Therefore the Lie algebra structure can be

carried over from $\kappa(\pi)$ to $\mathscr{F}(\pi, \pi)$. Denote the operation which thus arises in $\mathscr{F}(\pi, \pi)$ by $\{\cdot, \cdot\}$ we then have $[\chi_{\varphi_1}, \chi_{\varphi_2}] = \chi_{\{\varphi_1, \varphi_2\}}$, where $\chi_\psi \in \kappa$, $\chi_\psi \lrcorner U_1(\pi) = \psi$. If $\varphi = \chi \lrcorner U_1(\pi)$ we write $\mathfrak{Z}_\varphi$ instead of $\mathfrak{Z}_\chi$. In this notation

$$\{\varphi, \psi\} = \mathfrak{Z}_\varphi(\psi) - \mathfrak{Z}_\psi(\varphi) = (\mathfrak{Z}_\varphi - l_\varphi)(\psi). \tag{6.6.2}$$

Note that the solutions of the evolutionary equation $u_t = \Delta_\varphi(u)$ can be interpreted as trajectories of evolutionary derivations. For this reason evolutionary derivation may be intuitively understood as vector fields on the "manifolds" $\Gamma_{\mathrm{loc}}(\pi)$ while the velocity of evolution of the section $u \in \Gamma_{\mathrm{loc}}(\pi)$ generated by the flow corresponding to this vector field equals $u_t = \Delta_\varphi(u) = j(u)^*(\varphi)$. These remarks may be given an exact meaning up to infinitely small expressions of the second order with respect to t. Note that this also motivates the interpretation of $\kappa(N_m^\infty)$ as the set of all vector fields on the "manifold" of all n-dimensional submanifolds in N, and of Sym $\mathscr{Y}$ as the set of all vector fields on the "manifold" of all solutions of the equation $\mathscr{Y}$.

In the case $\pi = \mathbb{1}_M$ the Lie algebra $\mathscr{F}(\pi, \pi) = \mathscr{F}(\pi)$ possesses a subalgebra $\mathscr{F}_1(\pi)$ which coincides with the Lie algebra of contact vector fields on $J^1(\pi)$. The contact structure on $J^1(\pi)$ is then given by the form $U_1(\pi)$. The Lie algebra $\mathscr{F}_1(\pi)$ in its turn contains, as a subalgebra the Lie algebra of smooth functions on $T^*(M)$.

6.7. *Coordinates*

Suppose the bundle π is linear and $e_1, \ldots, e_m$ is a basis of sections of this bundle over the coordinate neighbourhood $V \subset M$. Then every section over V is given in the form $s = \sum u_i(x)\, e_i$, where $x = (x_1, \ldots, x_n)$ are coordinates in V. Then we obtain the diffeomorphism

$$f\colon \mathscr{U} = V \times \mathbb{R}^m \to E_\pi, \qquad f(x, u) = \sum u_i e_i|_x, \quad u = (u_1, \ldots, u_m),$$

and therefore the coordinate system x, p_τ^i in $J^\infty(\pi)$ (see 6.1). Elements of the module $\mathscr{F}(\pi, \pi)$ in $\pi_\infty^{-1}(V)$ can be written in the form $\varphi = \sum_{i=1}^m \varphi_i e_i$, where $\varphi_i \in \mathscr{F}(\pi)$ and we still denote $\pi_\infty^*(e_i)$ by e_i. If $f_1, \ldots, f_l$ is a basis of local sections of the bundle π' over V then similarly for $\psi \in \mathscr{F}(\pi, \pi')$ we have $\psi = \sum_{i=1}^l \psi_i f_i$ over V. Then

$$\mathfrak{Z}_\varphi^{\pi'} = \sum_{i,k,\tau} D_\tau(\varphi_i)\, f_k \frac{\partial^{(k)}}{\partial p_\tau^i},$$

where $\mathfrak{Z}_\varphi^{\pi'}\colon \mathscr{F}(\pi, \pi') \to \mathscr{F}(\pi, \pi')$ is the operator of evolutionary derivation

$$D_\tau = D_{i_1} \circ \cdots \circ D_{i_k}, \qquad \text{if} \quad \tau = (i_1,\dots, i_k),$$

$$D_i = \frac{\partial}{\partial x_i} + \sum_{\sigma, j} p^j_{\sigma i} \frac{\partial}{\partial p^i_\sigma}, \qquad \sigma i = (i, j_1, \dots, j_s), \quad \text{if } \sigma = (j_1, \dots, j_s),$$

$$\left(\frac{\partial^{(k)}}{\partial p^i_\tau}\right)(\psi) = \frac{\partial \psi_k}{\partial p^i_\tau}.$$

In particular,

$$\{\varphi, \psi\} = \sum_{i,k,\tau} \left(D_\tau(\varphi_i)\frac{\partial \psi_k}{\partial p^i_\tau} - D_\tau(\psi_i)\frac{\partial \varphi_k}{\partial p^i_\tau}\right) e_k, \qquad \varphi, \psi \in \mathscr{F}(\pi, \pi),$$

$$l_\psi = \sum_{i,k,\tau} \frac{\partial \psi_k}{\partial p^i_\tau} f_k D^{(i)}_\tau, \qquad \psi \in \mathscr{F}(\pi, \pi'),$$

where $D^{(i)}_\tau(\varphi) = D_\tau(\varphi_i)$.

7. Spencer Complexes and Green's Formula in $\mathscr{C}$-theory

In order to construct the nonlinear Lagrange formalism, just as in the linear case, we shall need a theory of adjoint operators constructed in agreement with the theory of Spencer complexes. It turns out that in the framework of $\mathscr{C}$-theory this can be done by practically the same means as before.

7.1. *The Operators $\hat{\Delta}$*

Suppose $\pi: E_\pi \to M$ is a smooth bundle, $\alpha: E_\alpha \to M$ and $\beta: E_\beta \to M$ are smooth vector bundles and $\Delta \in \mathrm{Diff}(\Gamma(\alpha), \Gamma(\beta))$. In this situation we can define the operator $\hat{\Delta}: \mathscr{F}(\pi, \alpha) \to \mathscr{F}(\pi, \beta)$, $\hat{\Delta}(\psi) = \varphi_{\Delta \circ \square}$, where $\psi = \varphi_\square \in \mathscr{F}(\pi, \alpha)$ (see 6.2). The operator $\hat{\Delta}$ increases filtration by s if $\Delta \in \mathrm{Diff}_s$ and is $\mathscr{C}$-differential (see [4]). For this reason (see 6.5). An important example is given by the operators $\hat{X} \in D(\mathscr{F}(\pi))$, where $X \in D(M)$. In a chart on $J^\infty(\pi)$ compatible with π the operators $D_i = \hat{\partial}/\partial x_i$ are of the form

$$D_i = \frac{\hat{\partial}}{\partial x_i} = \frac{\partial}{\partial x_i} + \sum_{\sigma, k} p^k_{\sigma i} \frac{\partial}{\partial p^k_\sigma}.$$

Put $D_\sigma = D_{i_1} \circ \cdots \circ D_{i_s}$ if $\sigma = (i_1, \dots, i_s)$ and note that $\hat{f} = f$, where $f \in C^\infty(M)$ is understood as the multiplication operator on f. In the view of the fact that $\widehat{\Delta_1 \circ \Delta_2} = \hat{\Delta}_1 \circ \hat{\Delta}_2$ it therefore follows that $\Delta = \sum_\sigma a_\sigma(x)\, \partial^{|\sigma|}/\partial x_\sigma \in \mathrm{Diff}\, C^\infty(M)$ implies $\hat{\Delta} = \sum_\sigma a_\sigma(x)\, D_\sigma$, while if Δ is given by the operator

matrix of scalar operators $\|\Delta_{ij}\|$, then $\hat{\Delta}$ is given by the matrix $\|\hat{\Delta}_{ij}\|$. This yields a complete description of these operators in coordinates. Also note the following property of the operators $\hat{\Delta}$:

$$l_{\hat{\Delta}(p)} = \hat{\Delta} \circ l_p, \qquad p \in \mathscr{F}(\pi, \alpha).$$

The operators of the form $\hat{\Delta}$ generate together with $\mathscr{F}(\pi)$ the algebra of operators $\mathscr{C}\,\mathrm{Diff}\,\mathscr{F}(\pi)$ (see [4]). Therefore the above shows that within a single chart the algebra $\mathscr{C}\,\mathrm{Diff}\,\mathscr{F}(A)$ is generated by the operators D_i and $\mathscr{F}(\pi)$. This in its turn shows that $\mathscr{C}D(A)$ and A generate $\mathscr{C}\,\mathrm{Diff}\,A$, where $A = \mathscr{F}(\pi)$, $\mathscr{F}_m(N)$, or $\mathscr{F}(\mathscr{Y})$. Indeed, in the first two cases this is clear from the fact that $J^\infty(\pi)$ or N_m^∞ may be covered by a system of charts, and in the last because $\mathscr{C}\,\mathrm{Diff}\,\mathscr{F}(\mathscr{Y})$ is the restriction of $\mathscr{C}\,\mathrm{Diff}\,\mathscr{F}(\pi)$ or $\mathscr{C}\,\mathrm{Diff}\,\mathscr{F}_m(N)$ to $\mathscr{Y}_\infty$. In fact, the ideal $\mathscr{I}_\infty$ of the equation $\mathscr{Y}_\infty$ in $A = \mathscr{F}(\pi)$ or $\mathscr{F}_m(N)$ is the radical of the ideal $\mathscr{C}\,\mathrm{Diff}\,A \cdot \mathscr{I}$, where $\mathscr{I}$ is the ideal of the submanifold $\mathscr{Y}$ in $J^k(\pi)$ or N_m^k. Therefore it is closed with respect to the action of $\mathscr{C}\,\mathrm{Diff}\,A$ and therefore every $\square \in \mathscr{C}\,\mathrm{Diff}\,A$ may be "lowered" onto $\mathscr{F}(\mathscr{Y}) = A/\mathscr{I}_\infty$.

7.2. *The Action of the Operators on the Forms in $\mathscr{C}$-Theory*

Suppose $A = \mathscr{F}(\pi)$ or $\mathscr{F}_m(N)$. In this case $\bar{\Lambda}^n$ is a one-dimensional locally free A-module. Locally means within of each chart $\{f_{(k)}\}$. Indeed, within any chart the module $\bar{\Lambda}^n$ may be identified with the module Λ_0^n of horizontal forms (with respect to this chart), see 6.4, while

$$\Lambda_0^n = A \otimes_{C^\infty(M)} \Lambda^n(M).$$

If $A = \mathscr{F}(\mathscr{Y})$ then the module $\bar{\Lambda}^n$ is also locally free, since $\bar{\Lambda}^n$ is the restriction of $\bar{\Lambda}^n(\mathscr{F}_m(N))$ (or $\bar{\Lambda}^n(\mathscr{F}(\pi))$ to $\mathscr{Y}_\infty$, whenever $\mathscr{Y}_\infty \subset N_m^\infty$ (or $J^\infty(\pi)$). In particular, $\bar{\Lambda}^i = 0$ if $i > n$ and therefore $\bar{d}(\bar{\Lambda}^n) = 0$.

PROPOSITION. *Suppose $A = \mathscr{F}(\pi)$, $\mathscr{F}_m(N)$, or $\mathscr{F}(\mathscr{Y})$. Then there exists a unique action of the operators $\Delta \in \mathscr{C}\,\mathrm{Diff}\,A$ on the forms $\bar{\omega} \in \bar{\Lambda}^n$, denoted by $\Delta[\bar{\omega}]$ and satisfying axioms* (1)–(4) *of* 1.1.

Proof. The proof of this statement is an exact repetition of the proof of Proposition 1.1, since, in accordance to 7.1 $\mathscr{C}\,\mathrm{Diff}\,A$ is generated by $\mathscr{C}D(A)$ and A. The only thing that has to be added is the notion of Lie derivative $X(\bar{\omega}) \in \bar{\Lambda}^k$, $X \in \mathscr{C}D(A)$, $\omega \in \bar{\Lambda}^k$, satisfying the formula $X(\bar{\omega}) = \bar{d}(X \lrcorner\, \bar{\omega}) + X \lrcorner\, \bar{d}\omega$. In order to obtain this notion, note first that the substitution operation $X \lrcorner\, \bar{\omega} = \overline{X \lrcorner\, \omega} \in \bar{\Lambda}^{k-1}$, $X \in \mathscr{C}D(A)$, where $\omega \in \Lambda^k$, $\bar{\omega} = \omega \bmod \mathscr{C}\Lambda^k$ and the horizontal line denotes factorization by $\bmod\,\mathscr{C}\Lambda^*$ is well defined. This is obvious in view of the relation $\mathscr{C}\Lambda^* = \mathscr{C}\Lambda^1 \wedge \Lambda^*$ since

$X \lrcorner \mathscr{C}\Lambda^1 = 0$ and therefore $X \lrcorner \mathscr{C}\Lambda^k \subset \mathscr{C}\Lambda^{k-1}$. Since $\mathscr{C}D(A) \subset D_{\mathscr{C}}(A)$ the definition $X(\bar{\omega}) = \overline{X(\omega)}$ is correct and

$$\overline{d(X \lrcorner \omega)} + \overline{X \lrcorner d\omega} = \bar{d}(\overline{X \lrcorner \omega}) + X \lrcorner \overline{d\omega}$$
$$= \bar{d}(X \lrcorner \bar{\omega}) + X \lrcorner \bar{d}\bar{\omega}. \quad \blacksquare$$

7.3. *Adjoint Operators in $\mathscr{C}$-Theory*

Further we assume that $A = \mathscr{F}(\pi)$, $\mathscr{F}_m(N)$, or $\mathscr{F}(\mathscr{Y})$. In any of these cases the theory of adjoint operators is constructed exactly in the same way as in Section 1. To be more precise, suppose $\Delta \in \mathscr{C}$ Diff $\bar{\Lambda}^n$ then the operator $\Delta^* \in \mathscr{C}$ Diff $\bar{\Lambda}^n$ is defined by the formula $\Delta^*(f) = \Delta_{\bar{\omega}}[f\bar{\omega}]$, where $\bar{\omega} \in \bar{\Lambda}^n$ is the local "volume form," i.e., the local generator of the A-module $\bar{\Lambda}^n$ while $\Delta(f) = \Delta_{\bar{\omega}}(f)\bar{\omega}$. This is a correct definition and this fact is checked just as in 1.2. The only thing which must be checked additionally is the fact that the operator Δ^* is $\mathscr{C}$-differential. But this is obvious if $\Delta = f$, $f \in A$. If $\Delta = \square \circ X$, $\square \in \mathscr{C}$ Diff $\bar{\Lambda}^n$, $X \in \mathscr{C}D(A)$, then

$$\Delta^*(f) = (\square \circ X)_{\bar{\omega}}\,[f\bar{\omega}] = (\square_{\bar{\omega}} \circ X)[f\bar{\omega}]$$
$$= -X(\square_{\bar{\omega}}[f\bar{\omega}]) = -X(\square^*(f)),$$

i.e., $\Delta^* = -X_{\bar{\Lambda}} \circ \square^*$, where $X_{\bar{\Lambda}}\colon \bar{\Lambda}^* \to \bar{\Lambda}^*$, $X_{\bar{\Lambda}}(\bar{\omega}) = X(\bar{\omega})$. But the composition of the operators $\Delta_1 \circ \Delta_2$ is $\mathscr{C}$-differential whenever the operators Δ_i are $\mathscr{C}$-differential. Therefore the $\mathscr{C}$-differentiality of the operator Δ^* is proved by induction over $\deg \Delta$ beginning with the representation

$$\Delta_{\bar{\omega}} = f + \sum_i X_1^i \circ \cdots \circ X_{s(i)}^i, \qquad f \in A,\ X_j^i \in \mathscr{C}D(A).$$

As in Section 1 suppose $\hat{Q} = \mathrm{Hom}_A(Q, \bar{\Lambda}^n)$ and for every $\Delta \in \mathscr{C}$ Diff$(P, \hat{Q})$ let us define the adjoint operator $\Delta^* \in \mathscr{C}$ Diff$(Q, \hat{P})$ by putting

$$\Delta^*(q, p) = \Delta(p, q)^*, \qquad p \in P,\ q \in Q.$$

Since the operators $\Delta^*(q, p)$ are $\mathscr{C}$-differential for $\forall p, q$, it follows that Δ^* itself is $\mathscr{C}$-differential.

PROPOSITION. *Propositions* 1.2 *and* 1.3 *remain valid if the operators appearing in their statement are replaced by $\mathscr{C}$-differential ones, the modules Λ^n by the modules $\bar{\Lambda}^n$ and* smbl_s *by* $\mathscr{C}$ smbl_s (see 7.4).

Proof. The proof is an exact repetition of the proofs of Propositions 1.2 and 2.3. $\blacksquare$

Suppose $A = \mathscr{F}_m(N)$ or $\mathscr{F}(\pi)$. Now for some chart (see 6.1) choose for the generator of $\bar{\Lambda}^n$ the form $\bar{\omega} = dx_1 \wedge \cdots \wedge dx_n$ and note, in accordance to

7.1 that every $\Delta \in \mathscr{C}\,\mathrm{Diff}\,A$ can be written in the form $\sum_\sigma a_\sigma D_\sigma$, $a_\sigma \in A$, then we can find, as in 1.4 the relation

$$\Delta^*_{\bar\omega} = \sum_\sigma (-1)^{|\sigma|} D_\sigma \circ a_\sigma, \qquad \text{if} \quad \Delta \in \mathscr{C}\,\mathrm{Diff}\,\bar{\Lambda}^n,\ \Delta_{\bar\omega} = \sum_\sigma a_\sigma D_\sigma.$$

If $\Delta \in \mathscr{C}\,\mathrm{Diff}_s(P, Q)$ as in 1.4 is given by the matrix $\|(\Delta_{ij})_{\bar\omega}\|$, $\Delta_{ij} \in \mathscr{C}\,\mathrm{Diff}\,\bar{\Lambda}^n$ it follows that Δ^* is given by the matrix $\|c_{kl}\|$, where $c_{kl} = (\Delta^*_{lk})_{\bar\omega}$.

7.4. *The $\mathscr{C}$-Hamiltonian Formalism*

Let us make the notion of $\mathscr{C}$-symbol $\mathscr{C}\,\mathrm{smbl}\,\Delta$ of a $\mathscr{C}$-differential operator Δ, which incidentally was mentioned in Proposition 7.3 more precisely suppose

$$\mathscr{C}\,\mathrm{smbl}_k(P, Q) = \mathscr{C}\,\mathrm{Diff}_k(P, Q)/\mathscr{C}\,\mathrm{Diff}_{k-1}(P, Q),$$
$$\mathscr{C}\,\mathrm{smbl}_k\,P = \mathscr{C}\,\mathrm{smbl}_k(A, P).$$

If $\Delta \in \mathscr{C}\,\mathrm{Diff}_k(P, Q)$ it follows that $\mathscr{C}\,\mathrm{smbl}_k\,\Delta = [\Delta \bmod \mathscr{C}\,\mathrm{Diff}_{k-1}(P, Q)] \in \mathscr{C}\,\mathrm{smbl}_k(P, Q)$. The module $\mathscr{C}\,\mathrm{smbl}(P, P) = \mathrm{Gr}\,\mathscr{C}\,\mathrm{Diff}(P, P) = \sum_{k \geqslant 0} \mathscr{C}\,\mathrm{smbl}_k(P, P)$ also happens to be a ring, and the ring $\mathscr{C}\,\mathrm{smbl}\,A$ is commutative. The composition operation of differential operators induces in $\mathscr{C}\,\mathrm{smbl}(P, Q)$ a left $\mathscr{C}\,\mathrm{smbl}(Q, Q)$-module structure and a right $\mathscr{C}\,\mathrm{smbl}(P, P)$-module structure. In particular, $\mathscr{C}\,\mathrm{smbl}\,P$ is a $\mathscr{C}\,\mathrm{smbl}\,A$-module.

Following [44], define the Poisson $\mathscr{C}$-bracket $\{\cdot, \cdot\}_{\mathscr{C}}$ on $\mathscr{C}\,\mathrm{smbl}\,A$ by putting

$$\{s_1, s_2\}_{\mathscr{C}} = \mathscr{C}\,\mathrm{smbl}_{i+j-1}[\Delta_1, \Delta_2],$$

where $\Delta_1 \in \mathscr{C}\,\mathrm{Diff}_i\,A$, $\Delta_2 \in \mathscr{C}\,\mathrm{Diff}_j\,A$, $s_1 = \mathscr{C}\,\mathrm{smbl}_i\,\Delta_1$, $s_2 = \mathscr{C}\,\mathrm{smbl}_j\,\Delta_2$. The $\mathscr{C}$-Poisson bracket supplies $\mathscr{C}\,\mathrm{smbl}\,A$ with a Lie algebra structures. Let us describe the bracket $\{\cdot, \}_{\mathscr{C}}$ in coordinates. Suppose $\Pi_i = \mathscr{C}\,\mathrm{smbl}_1\,D_i$. Then $\mathscr{C}\,\mathrm{smbl}_k\,\Delta = \sum_{|\sigma| = k} a_\sigma \Pi_\sigma$, where $\Delta = \sum_{|\sigma| \leqslant k} a_\sigma D_\sigma \in \mathscr{C}\,\mathrm{Diff}_k\,A$ and $\Pi^\sigma = \Pi_{i_1} \circ \cdots \circ \Pi_{i_s}$ whenever $\sigma = (i_1, \ldots, i_s)$. Thus $\mathscr{C}\,\mathrm{smbl}\,A$, within a single chart, is the polynomial ring in the variables Π_i with coefficients from A. It follows immediately from the definition that $\{a_1, a_2\}_{\mathscr{C}} = \{\Pi_i.\,\Pi_j\}_{\mathscr{C}} = 0$, $a_i \in A$, and $\{\Pi_i, a\}_{\mathscr{C}} = D_i(a)$, $a \in A$. Since the operator X_H, $X_H(h) = \{H, h\}_{\mathscr{C}}$, is a derivation of the algebra $\mathscr{C}\,\mathrm{smbl}\,A$ it follows from these relations that

$$\{H_1, H_2\}_{\mathscr{C}} = \sum_i \left(\frac{\partial H_1}{\partial \Pi_i} D_i(H_2) - \frac{\partial H_2}{\partial \Pi_i} D_i(H_1)\right),$$

where for $h = \sum_\sigma a_\sigma \Pi^\sigma$, $D_i(h) = \sum_\sigma D_i(a_\sigma)\,\Pi^\sigma$.

Remark. The homomorphism of filtered algebras $\mathrm{Diff}\,C^\infty(M) \to \mathscr{C}\,\mathrm{Diff}\,A$

for which $\Delta \mapsto \hat{\Delta}$ generates the homomorphism $\operatorname{smbl} C^{\infty}(M) \to \mathscr{C} \operatorname{smbl} A$ which is compatible with the Poisson brackets. Thus the Hamiltonian formalism on $T^*(M)$ (see [44]) can be included into the $\mathscr{C}$-Hamiltonian formalism.

7.5. *The* $\mathscr{C}$ Diff-*Spencer Complexes*

Now let us prove the fact that Spencer complexes in $\mathscr{C}$-theory are exact. To avoid a necessarily complex notation we shall further denote the composition $\Phi_1 \circ \cdots \circ \Phi_k$ of functors of functors of the differential calculus in $\mathscr{C}$-theory simply $\mathscr{C}\Phi_1 \circ \cdots \circ \Phi_k$ instead of $\mathscr{C}\Phi_1 \circ \cdots \circ \mathscr{C}\Phi_k$. For example, $\mathscr{C}D_i \operatorname{Diff}_k^+ = (\mathscr{C}D_i)(\mathscr{C}\operatorname{Diff}_k^+)$.

The proof of the fact that the Spencer $\mathscr{C}$ Diff-complexes are exact is based on the following general considerations. Suppose $P \mapsto S_k P$ (or $P \mapsto SP$) is the functor which assigns to every module P its Spencer complex $S_k P$ (or SP) in some category of differential operators. Suppose this functor is representable. Then its representative object is the complex constituted by the representative objects of the functors $\dot{D}_i \operatorname{Diff}_j^+$. Since the latter are the modules $\mathscr{T}^j(\Lambda^i)$ this complex is of the form

$$0 \to A \xrightarrow{j_k} \mathscr{T}^k \xrightarrow{S_{\mathscr{T}}} \mathscr{T}^{k-1}(\Lambda^1) \xrightarrow{S_{\mathscr{T}}} \cdots \xrightarrow{S_{\mathscr{T}}} \mathscr{T}^{k-i}(\Lambda^i) \xrightarrow{S_{\mathscr{T}}} \cdots. \qquad (\mathscr{T})$$

It follows from the explicit formula with the operators $S_{i,j}$ (see 2.1) that

$$S_{\mathscr{T}}(fj_s(\omega)) = fj_{s-1}(d\omega), \qquad \omega \in \Lambda^i,$$

(also see [13]). The following lemma is obvious.

LEMMA. *If the complex* $(\mathscr{T})$ *is exact, while the modules* $\mathscr{T}^j(\Lambda^i)$ *projective then the Spencer complexes* $S_k(P)$ *or* SP *in the category under consideration are exact.*

Let us apply this lemma to $\mathscr{C}$-theory. The Diff-Spencer complex in this theory, denote by $\mathscr{C}S_k P$ (or $\mathscr{C}SP = \mathscr{C}S_{\infty}P$) is of the form

$$0 \leftarrow P \xleftarrow{\Pi_k} \mathscr{C}\operatorname{Diff}_k P \xleftarrow{S_{1,k-1}} \mathscr{C}\dot{D}(\operatorname{Diff}_{k-1}^+ P)$$
$$\leftarrow \cdots \xleftarrow{S_{i,k-i}} \mathscr{C}\dot{D}_i(\operatorname{Diff}_{k-i}^+ P) \leftarrow \cdots.$$

According to 6.4 the representative object for $\mathscr{C}\dot{D}_i \operatorname{Diff}_s^+$ is $\overline{\mathscr{T}^s(\Lambda^i)}$. Therefore it suffices to show that the complex

$$0 \longrightarrow A \xrightarrow{\bar{i}_k} \bar{\mathscr{T}}^k \xrightarrow{\bar{S}_{\mathscr{T}}} \overline{\mathscr{T}^{k-1}(\Lambda^1)} \xrightarrow{\bar{S}_{\mathscr{T}}} \cdots$$

$$\xrightarrow{\bar{S}_{\mathscr{T}}} \overline{\mathscr{T}^{k-i}(\Lambda^i)} \xrightarrow{\bar{S}_{\mathscr{T}}} \cdots \qquad (\bar{\mathscr{T}}^k)$$

satisfies the assumptions of the lemma.

If $A = \mathscr{F}(\pi)$ or $\mathscr{F}(\mathscr{Y})$, where $\mathscr{Y}_\infty \subset J^\infty(\pi)$ the modules $\overline{\mathscr{T}^s(\Lambda^i)}$ are projective in view of the isomorphism $\bar{\Phi} = \Phi_0 = \Phi(M) \otimes_{C^\infty(M)} A$ (see 6.4). If $A = \mathscr{F}_m(N)$ or $A = \mathscr{F}(\mathscr{Y})$, where $\mathscr{Y}_\infty \subset N_m^\infty$ then the isomorphism mentioned above holds within every chart and therefore $\overline{\mathscr{T}^s(\Lambda^i)}$ is also projective.

Finally the $(\bar{\mathscr{T}}^k)$-complex is exact since the operators $\bar{S}_{\mathscr{T}}$ are A-homomorphisms while locally the $(\bar{\mathscr{T}}^k)$-complex may be represented in the form

$$(\mathscr{T}^k) = (\mathscr{T}^k(M)) \otimes_{C^\infty(M)} A,$$

where $(\mathscr{T}^k(M))$ is the Spencer $\mathscr{T}$-complex in the category of geometric modules over $C^\infty(M)$. Recall that $(\mathscr{T}^k(M))$ is an exact complex, see [4, 36]. Thus we have proved the following:

THEOREM. *If $A = \mathscr{F}(\pi)$, $\mathscr{F}_m(N)$, or $\mathscr{F}(\mathscr{Y})$ then the $(\bar{\mathscr{T}}^k)$-complex over A is exact just as are the complexes $\mathscr{C}S_kP$ or $\mathscr{C}SP$.*

7.6. *The $\mathscr{C}$-Analogue of the Theory in Section* 2

All the results and constructions of Section 2 with appropriate modifications may be carried over into $\mathscr{C}$-theory. This will be done below with certain comments.

The constructions and formulas of 2.1 are carried over into $\mathscr{C}$-theory word for word as long as the operators in them are assumed $\mathscr{C}$-differential and respectively the functors of the differential calculus Φ are replaced by $\mathscr{C}\Phi$.

Poincare duality in $\mathscr{C}$-theory, which is needed to make the constructions of 2.2 meaningful is essentially an isomorphism $\Pi\colon \mathscr{C}D_i(\bar{\Lambda}^n) \to \bar{\Lambda}^{n-i}$ defined by the relation

$$\Pi(X_1 \wedge \cdots \wedge X_i \otimes \bar{\omega}) = X_1 \lrcorner (\cdots (X_i \lrcorner \bar{\omega}) \cdots).$$

Here $X_i \in \mathscr{C}D(A)$, $\bar{\omega} \in \bar{\Lambda}^n$ is the local volume form while the operation $Y \lrcorner \rho$, $Y \in \mathscr{C}D(A)$, $\rho \in \bar{\Lambda}^k$ is understood in the sense of 7.2. The homomorphism Π is an isomorphism since within any affine chart we have $\bar{\Lambda}^i = \Lambda^i(M) \otimes_{C^\infty(M)} A$.

The complex

$$0 \longleftarrow \bar{\Lambda}^n \xleftarrow{\mu} \mathscr{C}\,\mathrm{Diff}_k^+\,\bar{\Lambda}^n \xleftarrow{\mathscr{S}} \mathscr{C}\,\mathrm{Diff}_{k-1}^+\bar{\Lambda}^{n-1} \xleftarrow{\mathscr{S}} \cdots$$

$$\xleftarrow{\mathscr{S}} \mathscr{C}\,\mathrm{Diff}_{k-n}^+ A \longleftarrow 0$$

is defined exactly in the same way: $\mathscr{S}(\nabla)=\bar{d}\circ\nabla$, $\mu(\nabla)=\nabla^*(1)$. It is only necessary to note that the operator $\bar{d}\circ\nabla$ is also $\mathscr{C}$-differential. But this follows immediately from the fact that $\bar{d}$ is $\mathscr{C}$-differential (see (6.4).

After these remarks the analogues of the isomorphism ψ

$$\psi\colon \mathscr{C}\dot{D}_i\,\mathrm{Diff}_s^+\,\bar{\Lambda}^n\to\mathscr{C}\,\mathrm{Diff}_s^+\,\bar{\Lambda}^{n-i}$$

together with the statements and proofs of Theorems 2.2 and Corollary 2.2 can be carried over word for word to $\mathscr{C}$-theory just as the construction of the complexes $\mathscr{S}_kP$ and $\mathscr{S}_{k,l}PQ$. For the sake of brevity we shall retain the same notation for these complexes in $\mathscr{C}$-theory. Theorem 3.1, together with the proof, can also be immediately carried over to $\mathscr{C}$-theory.

Proposition 2.3 together with the calculations proves it remains valid for the de Rham $\bar{d}$-complex

$$0\to A\xrightarrow{\bar{d}}\bar{\Lambda}^1\xrightarrow{\bar{d}}\bar{\Lambda}^2\xrightarrow{\bar{d}}\cdots\xrightarrow{\bar{d}}\bar{\Lambda}^n\to 0$$

and the Spencer $\overline{\mathscr{S}_kP}$ complex

$$\begin{aligned}0\longrightarrow P\xrightarrow{j_k}\overline{\mathscr{J}^k(P)}\xrightarrow{\bar{\Sigma}_{0,k}}\overline{\mathscr{J}^{k-1}(P)}\otimes\bar{\Lambda}^1\\ \xrightarrow{\bar{\Sigma}_{1,k-1}}\cdots\xrightarrow{\bar{\Sigma}_{n-1,n-k-1}}\overline{\mathscr{J}^{k-n}(P)}\otimes\bar{\Lambda}^n\longrightarrow 0.\end{aligned}$$

Green's $\mathscr{C}$-formula (i.e., the analogue of Green's formula in $\mathscr{C}$-theory) in view of the above requires for its proof only the existence of the splitting $\lambda\colon\mathscr{C}\,\widetilde{\mathrm{Diff}}_k\,\bar{\Lambda}^n\to\mathscr{C}\dot{D}(\mathrm{Diff}_k^+\,\bar{\Lambda}^n)$. The projectivity of the A-modules $\mathscr{C}\,\mathrm{Diff}\,\bar{\Lambda}^i$ and therefore of the modules $\mathscr{C}\,\mathrm{Diff}_k\,\bar{\Lambda}^i$ suffices for this and follows from the fact that within an affine chart we have

$$\mathscr{C}\,\mathrm{Diff}\,\bar{\Lambda}^i=\mathrm{Diff}\,\Lambda^i(M)\otimes_{C^\infty(M)}A.$$

This isomorphism is established by the map $\Delta\otimes a\mapsto a\hat{\Delta}$, where $a\in A$. $\Delta\in\mathrm{Diff}\,\Lambda^i(M)$ (see 7.1). In view of its importance, we write out Green's formula in $\mathscr{C}$-theory explicitly

$$\langle\Delta(p),q\rangle-\langle p,\Delta^*(q)\rangle=\bar{d}\,Ж_\lambda(\Delta(p,q)),\qquad p\in P,q\in Q,\Delta\in\mathscr{C}\,\mathrm{Diff}\,(P,\hat{Q}).$$

Subsection 2.5 is repeated in $\mathscr{C}$-theory without notable changes. The coordinate notation of the operator $Ж_\lambda$ and of the splitting λ within an affine chart under the condition $\bar{\omega}=dx_1\wedge\cdots\wedge dx_n\in\bar{\Lambda}^n$, for the same reasons as in 2.6 is given by the formulas

$$\lambda(\Delta)_{\bar{\omega}}=\sum_\sigma a_\sigma D_{\bar{\sigma}}\otimes D_{i(\sigma)},\qquad \Delta=\sum_{|\sigma|>0}a_\sigma D_\sigma,\quad a_\sigma\in A$$

(see 7.1)

$$Ж_\lambda(\Delta) = \sum_{|\sigma|>0} (-1)^{|\bar{\sigma}|} D_{\bar{\sigma}}(a_\sigma)\,\bar{\omega}_{i(\sigma)}, \qquad \bar{\omega}_i = (-1)^{i-1} D_i \lrcorner\, \bar{\omega}.$$

Further, when we refer to the $\mathscr{C}$-analogues of the "usual" results and constructions discussed in the present section we shall add to the "ordinary" reference the letter $\mathscr{C}$, e.g., as in Theorem 2.2 $\mathscr{C}$.

7.7. *Transformations*

Every morphism $F: A_1 \to A_2$ in DE (see 6.4) naturally induces a homomorphism $F_{\bar{\Lambda}} = F: \bar{\Lambda}^*(A_1) \to \bar{\Lambda}^*(A_2)$. If F is an isomorphism in DE then, moreover, it generates under the same assumptions and the same formulas as in 1.5 the transformations F_{Diff}: $\mathrm{Diff}(P_1, Q_1) \to \mathrm{Diff}(P_2, Q_2)$. Here obviously $F_{\mathrm{Diff}}(\mathscr{C}\,\mathrm{Diff}(P_1, Q_1) \subset \mathscr{C}\,\mathrm{Diff}(P_2, Q_2)$ so that we can put $F_{\mathscr{C}\,\mathrm{Diff}} = F_{\mathrm{Diff}}|_{\mathscr{C}\,\mathrm{Diff}}$. Therefore if in the definitions and formulas of subsections 1.5 and 2.7, concerning finite transformations, we carry out the substitutions

$$\Lambda^* \to \bar{\Lambda}^*,\ \mathrm{Diff} \to \mathscr{C}\,\mathrm{Diff},\ F_\Lambda \to F_{\bar{\Lambda}},\ F_{\mathrm{Diff}} \to F_{\mathscr{C}\,\mathrm{Diff}},$$

then these definitions and formulas remain valid. It is interesting to note, however, that in this situation for F we can take the "restriction maps" and not only isomorphisms. To be more precise, suppose the equation $\mathscr{Y}$ is a corollary of the equation $\mathscr{Y}'$, i.e., $\mathscr{Y}'_\infty \overset{i}{\hookrightarrow} \mathscr{Y}_\infty$ (in particular, we do not exclude the cases $\mathscr{Y}_\infty = N^\infty_m$ and $\mathscr{Y}_\infty = J^\infty(\pi)$) or $\mathscr{I}_{\mathscr{Y}'} \supset \mathscr{I}_{\mathscr{Y}}$ (see 6.4). Suppose

$$A = \mathscr{F}(\mathscr{Y}), \qquad A' = \mathscr{F}(\mathscr{Y}'), \qquad F = i^*: A \to A'$$

and let $F_{\bar{\Lambda}}$: $\bar{\Lambda}^*(A) \to \bar{\Lambda}^*(A')$ be the map of forms which corresponds F. Since the ideal $\mathscr{I} = \mathscr{I}_{\mathscr{Y}'}/\mathscr{I}_{\mathscr{Y}} \subset A$ is stable with respect to the action of $\mathscr{C}\,\mathrm{Diff}\,A$ it follows from $F_{\bar{\Lambda}}(\bar{\omega}) = 0$, $\bar{\omega} \in \bar{\Lambda}^*(A)$ that $F_{\bar{\Lambda}}(X(\bar{\omega})) = 0$ for all $X \in \mathscr{C}D(A)$. For the same reason, the restriction map $F_{\mathscr{C}\,\mathrm{Diff}}$: $\mathscr{C}\,\mathrm{Diff}\,A \to \mathscr{C}\,\mathrm{Diff}\,A'$ is well defined and is an epimorphism:

$$F_{\mathscr{C}\,\mathrm{Diff}}(\Delta)(F(\varphi)) = F(\Delta(\varphi)), \qquad \Delta \in \mathscr{C}\,\mathrm{Diff}\,A,\ \varphi \in A.$$

It follows from these two facts that the operation $[\cdot]_0$

$$\Delta[\bar{\omega}]_0 = F_{\bar{\Lambda}}(\Box[\bar{\rho}]), \qquad \Box \in \mathscr{C}\,\mathrm{Diff}\,A, \qquad \Delta = F_{\mathscr{C}\,\mathrm{Diff}}(\Box),$$
$$\bar{\rho} \in \bar{\Lambda}^n(A),\ \bar{\omega} = F_{\bar{\Lambda}}(\bar{\rho}),$$

on $\mathscr{Y}'_\infty$ is well defined and satisfies the axioms of action. Since the action is

unique $[\cdot]_0 = [\cdot]$, i.e., the operation $[\cdot]$ is compatible with restriction maps. This in turn implies

$$F_{\mathscr{C}\,\mathrm{Diff}}(\Box^*) = F_{\mathscr{C}\,\mathrm{Diff}}(\Box)^*, \qquad \Box \in \mathscr{C}\,\mathrm{Diff}\,A.$$

The homomorphism F, as we see from obvious functoriality considerations, generates the Spencer $\mathscr{C}$-complex homomorphisms over A into the Spencer $\mathscr{C}$-complexes over A' as well as homomorphisms of complexes

$$\mathscr{S}_k P \to \mathscr{S}_k P', \qquad \mathscr{S}_{k,l} PQ \to \mathscr{S}_{k,l} P'Q'$$

in $\mathscr{C}$-theory where $P' = P|_{\mathscr{Y}'_\infty}$, $Q' = Q|_{\mathscr{Y}'_\infty}$. In particular, in view of Theorem 2.2 $\mathscr{C}$, Green's formula for the operator $\Box \in \mathscr{C}\,\mathrm{Diff}(P, Q)$ over A possesses a restriction to $\mathscr{Y}'_\infty$ and this restriction is Green's formula for

$$\Delta = F_{\mathscr{C}\,\mathrm{Diff}}(\Box) \in \mathscr{C}\,\mathrm{Diff}(P', Q').$$

Now consider in the role of $\mathscr{Y}' \subset N^k_m$ (or $J^k(\pi)$) submanifolds of the form $L^{(k)} = \mathrm{im}\, j_k(L)$ (or $\mathrm{im}\, j_k(\sigma)$, $\sigma \in \Gamma_{\mathrm{loc}}(\pi)$) (see 6.1). Then, independently of k we have $\mathscr{Y}'_\infty = \mathrm{im}\, j(L)$ (or $\mathrm{im}\, j(\sigma)$). For this reason, in the role of A' we can take $C^\infty(L)$ (or $C^\infty(\mathscr{U})$, where $\mathscr{U}$ is the domain of σ) while for F we can choose $j(L)^*$ (or $j(\sigma)^*$). Thus, adjoint operator theory and the related formulas which were discussed above, are compatible with homomorphisms of the form $F = j(L)^*$ or $j(\sigma)^*$. Note that in this case $\mathscr{C}\Phi = 0$ for all $\Phi = \Phi(\mathscr{Y}'_\infty)$ and therefore $\bar{\Phi} = \Phi$. In view of this homomorphism $F = j(L)^*$ ($= j(\sigma)^*$) induces the homomorphism

$$F_{\bar{\Lambda}} \colon \bar{\Lambda}^*(A) \to \Lambda^*(L), \qquad (F_{\bar{\Lambda}} \colon \bar{\Lambda}^*(A) \to \Lambda^*(\mathscr{U})).$$

In other words, $L \subset N$ (or $\mathscr{U} \subset M$) may be viewed as an object of DE if we put $\mathscr{C} \equiv 0$. Then $j(L)^*$ (or $j(\sigma)^*$) is a morphism in DE.

The theory of infinitesimal transformations in the situation considered is actually somewhat richer than in the "ordinary" case. Indeed, all the definitions and formulas from subsections 1.5 and 2.7, together with their proofs, remain valid, as long as we take for the infinitesimal transformations the "field" $X \in \mathscr{C}D(A)$. However, it is possible to extend the class of infinitesimal transformations by taking "fields" from $D_{\mathscr{C}}(A)$ and preserve all the appropriate formulas. The proof of these formulas now necessitates other arguments, than those carried out in 1.5 and 2.7 and will be given below. For example, if $X \in D_{\mathscr{C}}(A) \setminus \mathscr{C}D(A)$ then the relation $[X, \Delta]^* = [X, \Delta^*]$ cannot be proved by writing out the commutator on the left-hand side and using the properties of the operation $*$ since the operator X^* in this case is meaningless. Therefore we now assume that $X \in D_{\mathscr{C}}(A)$ and the definition of

infinitesimal transformations of corresponding objects remain the same as in subsections 1.5 and 2.7. We stress here that according to the definition of $D_{\mathscr{C}}(A)$ the relation

$$X(\bar{\omega}) = \overline{X(\omega)}, \qquad \omega \in \Lambda^k,\ \bar{\omega} = (\omega \bmod \mathscr{C}\Lambda^k) \in \bar{\Lambda}^k$$

gives a correct definition of the "Lie derivative" $X(\bar{\omega})$ of any form from $\bar{\Lambda}^*$ along the field $X \in D_{\mathscr{C}}(A)$.

(1) *Proof of the formula* $\Delta[\bar{\omega}]_X \equiv 0$ *or* $X(\Delta[\bar{\omega}]) = [X, \Delta][\bar{\omega}] + \Delta[X(\bar{\omega})]$. If $\Delta \in A = \mathscr{C}\operatorname{Diff}_0 A$ then $[X, \Delta] = X(\Delta) \in A$ and the relation we need becomes obvious. Suppose $\Delta \in \mathscr{C}D(A)$. Then

$$[X, \Delta][\bar{\omega}] = -[X, \Delta](\bar{\omega}) = \Delta(X(\bar{\omega})) - X(\Delta(\bar{\omega})),$$
$$\Delta[X(\bar{\omega})] = -\Delta(X(\bar{\omega}), \qquad X(\Delta[\bar{\omega}]) = -X(\Delta(\bar{\omega})).$$

Substituting these expressions in the formula which is to be proved, we see that it is valid. Suppose finally $\Delta = \Box \circ \nabla$ and assume that $\Box$ and ∇ satisfy the formula in question. Let us show that it is also valid for Δ.

First,

$$X(\Delta[\bar{\omega}]) = X((\Box \circ \nabla)[\bar{\omega}]) = X(\nabla[\Box[\bar{\omega}]])$$
$$\Delta[X(\bar{\omega})] = (\Box \circ \nabla)[X(\bar{\omega})] = \nabla[\Box[X(\bar{\omega})]].$$

Second, using the relation $[X, \Box \circ \nabla] = \Box \circ [X, \nabla] + [X, \Box] \circ \nabla$ we obtain $[X, \Delta][\bar{\omega}] = [X, \nabla][\Box[\bar{\omega}]] + \nabla[[X, \Box][\bar{\omega}]]$. Since the formula which we are proving is valid for the operators $\Box$ and ∇, we have

$$[X, \nabla][\Box[\bar{\omega}]] = X(\nabla[\Box[\bar{\omega}]]) - \nabla[X(\Box[\bar{\omega}])],$$
$$[X, \Box][\bar{\omega}] = X(\Box[\bar{\omega}]) - \Box[X(\bar{\omega})]$$

hence

$$[X, \Delta][\bar{\omega}] = X(\nabla[\Box[\bar{\omega}]]) - \nabla[\Box[X(\bar{\omega})]].$$

Putting together the equalities we have obtained we get the necessary result. Finally, noticing that $\mathscr{C}\operatorname{Diff} A$ is generated by A and $\mathscr{C}D(A)$ we see that the facts proved above establish our formula by induction over $\operatorname{ord} \Delta$. ∎

(2) *The formula* $[X, \Delta]^* = [X, \Delta^*]$, $\Delta \in \mathscr{C}\operatorname{Diff}\bar{\Lambda}^n$. Suppose $\bar{\omega} \in \bar{\Lambda}^n$ is a local volume form. It may be checked directly that

$$[X, \Delta]_{\bar{\omega}} = [X, \Delta_{\bar{\omega}}] + \operatorname{div}_{\bar{\omega}} X \cdot \Delta_{\bar{\omega}},$$

where $X(\bar{\omega}) = \mathrm{div}_{\bar{\omega}} X \cdot \bar{\omega}$. Then

$$[X, \Delta]^* (f) = [X, \Delta]_{\bar{\omega}} [f\bar{\omega}] = [X, \Delta_{\bar{\omega}}][f\bar{\omega}] + \Delta_{\bar{\omega}}[\mathrm{div}_{\bar{\omega}} X \cdot f\bar{\omega}].$$

But according to what we have already proved

$$[X, \Delta_{\bar{\omega}}][f\bar{\omega}] = X(\Delta_{\bar{\omega}}[f\bar{\omega}]) - \Delta_{\bar{\omega}}[X(f\bar{\omega})]$$

so that

$$[X, \Delta]^* (f) = X(\Delta_{\bar{\omega}}[f\bar{\omega}]) - \Delta_{\bar{\omega}}[X(f)\bar{\omega}].$$

Since $X(\Delta_{\bar{\omega}}[f\bar{\omega}]) = X(\Delta^*(f))$ and $\Delta_{\bar{\omega}}[X(f)\bar{\omega}] = \Delta^*(X(f))$ we obtain the necessary result. ∎

(3) *The formula* $[X, \Delta]^* = [X, \Delta^*]$, $\Delta \in \mathscr{C}\,\mathrm{Diff}(P, \hat{Q})$. Recall that here we assume given the operators $X_P \in \mathrm{Der}\, P$ and $X_Q \in \mathrm{Der}\, Q$ (which covers X) and

$$[X, \Delta] = \hat{X}_Q \circ \Delta - \Delta \circ X_P, \qquad [X, \Delta^*] = \hat{X}_P \circ \Delta^* - \Delta^* \circ X_Q,$$

where for $Y \in \mathrm{Der}\, R$ $\hat{Y} \in \mathrm{Der}\, \hat{R}$ denotes the operator $\hat{r} \mapsto \hat{Y}(\hat{r}) = [Y, \hat{r}]$: $R \to \bar{\Lambda}^n$. Further the indices P and Q for the operators X_P, X_Q are omitted. $[X, \Delta]^* \ (q, p) = ([X, \Delta](p, q))^* = (q^* \circ [X, \Delta] \circ p)^*$ (see 1.3). Using the following obvious relations

$$\begin{aligned} [X, q^* \circ \Delta \circ p] &= [X, q^*] \circ \Delta \circ p + q^* \circ [X, \Delta] \circ p + q^* \circ \Delta \circ [X, p] \\ &= X(q)^* \circ \Delta \circ p + q^* \circ [X, \Delta] \circ p + q^* \circ \Delta \circ X(p) \end{aligned}$$

we get $q^* \circ [X, \Delta] \circ p = [X, q^* \circ \Delta \circ p] - X(q)^* \circ \Delta \circ p - q^* \circ \Delta \circ X(p)$. But since $q^* \circ \Delta \circ p \in \mathscr{C}\,\mathrm{Diff}\, \bar{\Lambda}^n$ and therefore, from what we have already proved $[X, q^* \circ \Delta \circ p]^* = [X, p^* \circ \Delta^* \circ q]$ we see that

$$\begin{aligned} (q^* \circ [X, \Delta] \circ p)^* &= [X, p^* \circ \Delta^* \circ q] - p^* \circ \Delta^* \circ X(q) - X(p)^* \circ \Delta^* \circ q \\ &= p^* \circ [X, \Delta^*] \circ q = [X, \Delta^*](q, p). \quad ∎ \end{aligned}$$

(4) The formulas $X(\mu) = 0$, $X(\text{Ж}_\lambda) = \bar{d} \circ v$ are established just as in subsection 2.7 by using the relations proved above.

8. The Nonlinear Lagrangian Formalism

This section is mainly devoted to the description, in appropriate terms, of the main notions of the nonlinear Lagrangian formalism. This will enable us in the sequel to place the Lagrangian formalism within the framework of the

spectral sequence and in this section to give a general theory of transversality conditions and establish a most general version of Noether's theorem.

8.1. *Densities of Lagrangians*

Sections belonging to $\Gamma_{\mathrm{loc}}(\pi)$ or n-dimensional submanifolds in $N = N^{n+m}$ will be called admissible for the given variational problem, if they satisfy all the supplementary conditions which appear in the statement of the problem, e.g., boundary conditions or constraints. The set of admissible sections will be denoted by $\Gamma_{\mathrm{adm}}(\pi)$ (resp. $\mathscr{M}^n_{\mathrm{adm}}(N)$) and we shall put

$$B' = \{ j(\sigma)(x) \mid \sigma \in \Gamma_{\mathrm{adm}}(\pi),\ x \in \operatorname{int} \mathscr{U},\ \mathscr{U} \text{ is the domain of } \sigma \}$$

(resp. $B' = \{ j(V)(x) \mid V \in \mathscr{M}^n_{\mathrm{adm}}(N),\ x \in \operatorname{int} V\}$. The boundary conditions (as will be explained in detail) determine in $J^\infty(\pi)$ (resp. N^∞_m) the "submanifold" of boundary conditions ∂B, which is characterized by the property

$$j(\sigma)(\partial \mathscr{U}) \subset \partial B, \qquad \sigma \in \Gamma_{\mathrm{adm}}(\pi),$$

(resp. $j(V)(\partial V) \subset \partial B$, $V \in \mathscr{M}^n_{\mathrm{adm}}(N)$). Suppose $B = B' \cup \partial B$ and

$$\bar{\Lambda}^*(B, \partial B) = \{ \bar{\omega} \in \bar{\Lambda}^*(B) \mid \bar{\omega}|_{\partial B} = 0 \}, \qquad \bar{\Lambda}^*(B) = \Lambda^*|_B.$$

Usually an expression of the form $\mathscr{L} = \int L\, dx_1 \cdots dx_n$ is called a Lagrangian, while the function L depending on the functions $u_1, \dots, u_m$ and their derivatives is called the density of the Lagrangian. Since, nevertheless, the object that is integrated must be an n-dimensional form, and not a function, it is more invariant to understand by the density of the Lagrangian the form $L\, dx_1 \wedge \cdots \wedge dx_n$ which may be interpreted as a horizontal form on $J^\infty(\pi)$. The above if we take into consideration that Λ^n_0 on $J^\infty(\pi)$ is naturally identified with $\bar{\Lambda}^n$ (see 6.5), motivates the following:

DEFINITION. The density of a Lagrangian is a form $\omega \in \bar{\Lambda}^n(B)$.

Remark. Usually $\omega \in \Lambda^n$ is given and its restriction to B is considered.

8.2. *Lagrangians*

The choice of the density of a Lagrangian ω determines a functional on $\Gamma_{\mathrm{adm}}(\pi)$ (resp. $\mathscr{M}^n_{\mathrm{adm}}(N)$) $\sigma \mapsto \int_{\mathscr{U}} j(\sigma)^* (\omega)$, $\sigma \in \Gamma_{\mathrm{adm}}(\pi)$, or, respectively $V \mapsto \int_V j(V)^* (\omega)$, $V \in \mathscr{M}^n_{\mathrm{adm}}(N)$, where $\mathscr{U}$ is the domain of σ while $j(\sigma)^*$ (resp. $j(V)^*$) is understood as a homomorphism from $\bar{\Lambda}^n(B)$ to $\Lambda^n(\mathscr{U})$ (resp. $(\Lambda^n(V))$, see 7.7. Here we assume the domain $\mathscr{U}$ (resp. the manifold V) are oriented.

The density $\omega \in \Lambda^n(B)$ is called trivial, if it determines the zero functional. In particular, if $\psi \in \bar{\Lambda}^{n-1}(B, \partial B)$ then the density $\bar{d}\psi$ is trivial in view of the

Stokes formula. Thus, the functional corresponding to the density $\omega \in \bar{\Lambda}^n(B) = \bar{\Lambda}^n(B, \partial B)$ depends on the cohomology class $\langle \omega \rangle \in \bar{H}^n(B, \partial B)$ ($\bar{H}^i$ denotes the cohomology of the Rham $\bar{d}$-complex). Taking into consideration the fact that integration is an operation which assigns to any form of the highest dimension its cohomology class, is natural to state the following

DEFINITION. The Lagrangian of a variational problem is a cohomology class $\mathscr{L} \in \bar{H}^n(B, \partial B)$.

Since, as we already pointed out in 7.7, the homomorphism $j(\sigma)^*$ (resp. $j(V)^*$) is a morphism in DE; it induces a homomorphism of $\bar{H}^n(B, \partial B)$ into $H^n(\mathscr{U}, \partial\mathscr{U})$ (resp. $H^n(V, \partial V)$) also denoted $j(\sigma)^*$ (resp. $j(V)^*$), so that the functional discussed above can be represented in the form

$$\sigma \longmapsto j(\sigma)^*\,(\mathscr{L}) \qquad (\text{resp. } V \longmapsto j(V)^*\,(\mathscr{L}).$$

The variational problem consists in finding the extremum of this functional.

Sometimes one must consider Lagrangian of the form

$$\mathscr{L} = \mathscr{L}' + \mathscr{L}'', \qquad \mathscr{L}' \in \bar{H}^n(B, \partial B), \qquad \mathscr{L}'' \in \bar{H}^{n-1}(\partial B),$$

to which correspond the functionals

$$\sigma \longmapsto j(\sigma)^*\,(\mathscr{L}') + (j(\sigma)|_{\partial\mathscr{U}})^*\,(\mathscr{L}''),$$

respectively,

$$V \longmapsto j(V)^*\,(\mathscr{L}) + (j(V)|_{\partial V})^*\,(\mathscr{L}'').$$

The domain $\mathscr{U}$ and its boundary $\partial\mathscr{U}$ (resp. V and ∂V) are supplied with compatible orientation).

8.3. *The First Variation Formula and Euler's Operators*

In this subsection we assume that $B' \subset J^\infty(\pi)$ is open. This means, in particular, that no conditions of the differential equation type are imposed on the sections which are varied. Suppose $\sigma \in \Gamma_{\mathrm{adm}}(\pi)$, $\mathscr{L} = \langle \omega \rangle$, $\omega \in \bar{\Lambda}^n(B)$. Consider some $\mathfrak{Z}_\chi$, $\chi \in \kappa(\pi)$. Then the velocity of change of the Lagrangian $\mathscr{L}$ under the evolution determined by this field (see 6.6) equals $\mathfrak{Z}_\chi(\mathscr{L}) = \langle \mathfrak{Z}_\chi(\omega) \rangle = \langle l_\omega(\chi) \rangle$ while the velocity $\dot{\mathscr{L}}_\sigma$ of change of the corresponding functional on σ equals $j(\sigma)^*\,\mathfrak{Z}_\chi(\mathscr{L}) = \int_{\mathscr{U}} j(\sigma)^*\,(l_\omega(\chi))$, where $\mathscr{U}$ is the domain of σ. According to Green's $\mathscr{C}$-formula we have

$$l_\omega(\chi) = \langle \chi, l_\omega^*(1) \rangle + \bar{d}\,\mathfrak{K}_\lambda(l_\omega(\chi, 1))$$

so that

$$\dot{\mathscr{L}}_\sigma = \int_{\mathscr{U}} j(\sigma)^* \langle \chi, l^*_\omega(1)\rangle + \int_{\partial\mathscr{U}} j(\sigma)^* Ж_\lambda(l_\omega(\chi, 1))$$

$$= \int_{\mathscr{U}} \langle j(\sigma)^* (\chi), j(\sigma)^* (l^*_\omega(1)\rangle + \int_{\partial\mathscr{U}} Ж_{\lambda(\sigma)}(l_{\omega,\sigma}(\chi, 1)),$$

where $\lambda(\sigma)$, $l_{\omega,\sigma}(\chi, 1)$ is the transformation of the splitting λ and the operator $l_\omega(\chi, 1)$, respectively, under the morphism $j(\sigma)^*$ in DE (see (7.7). In the case when π is linear and $3_\chi = 3_\varphi$, $\varphi = \chi \lrcorner U_1(\pi)$, the equality obtained, which is further called the first variation formula, may be rewritten in the form

$$\dot{\mathscr{L}}_\sigma = \int_{\mathscr{U}} \langle \Delta_\varphi(\sigma), j(\sigma)^* l^*_\omega(1)\rangle + \int_{\partial\mathscr{U}} Ж_{\lambda(\sigma)}(l_{\omega,\sigma}(\varphi, 1)).$$

Any section ψ of the bundle π over $\mathscr{U}$ possessing a compact support contained in int $\mathscr{U}$ may be represented in the form $\psi = \Delta(\sigma)$, where the support of the operator Δ also is contained in int $\mathscr{U}$. In this case

$$l_{\omega,\sigma}(\varphi, 1) = l_{\omega,\sigma} \circ j(\sigma)^* (\varphi) = l_{\omega,\sigma} \circ \Delta_\varphi(\sigma)$$

which is equal zero on $\partial\mathscr{U}$, where $\varphi = \varphi_\Delta$ and therefore $\dot{\mathscr{L}}_\sigma = \int (\psi, j(\sigma)^* l^*_\omega(1))$, $\psi = \Delta_\varphi(\sigma)$. Hence, in view of the Dubois–Raymond lemma the extremality condition $\dot{\mathscr{L}}_\sigma = 0$ implies $j(\sigma)^* l^*_\omega(1) = 0$. In other words, σ is a solution of the equation $l^*_\omega(1) = 0$, i.e., $\Box(\sigma) = 0$, where $\varphi_\Box = l^*_\omega(1) \in \Lambda^n_0 = \bar{\Lambda}^n$.

If the bundle π is nonlinear, this conclusion remains valid since every section $\sigma \in \Gamma_{\text{loc}}(\pi)$ may be surrounded by a subbundle with linear structures. Thus we have obtained the following:

THEOREM. *A necessary condition of the extremality of section $\sigma \in \Gamma_{\text{adm}}(\pi)$ for the Lagrangian $\mathscr{L} \in \bar{H}^n(B, \partial B)$, where B' is open in $J^\infty(\pi)$ is the relation $j(\sigma)^* l^*_\omega(1) = 0$, i.e., σ is a solution of the equation $l^*_\omega(1) = 0$.*

Note that the form $l^*_\omega(1) \in \bar{\Lambda}^n(B)$ is uniquely determined by the Lagrangian $\mathscr{L}$. Indeed $l_{\bar{d}\psi} = \bar{d} \circ l_\psi$. Therefore,

$$l^*_{\omega + \bar{d}\psi}(1) = l^*_\omega(1) + (\bar{d} \circ l_\psi)^* (1) = l^*_\omega(1) \pm l^*_\psi(\bar{d}(1)) = l^*_\omega(1)$$

since by Proposition 2.3 $\mathscr{C}$ we have $\bar{d}^* = \pm\bar{d}$.

The equation $l^*_\omega(1)$ will be called the Euler–Lagrange equation of the given variational problem. Written in local coordinates, it coincides with the classical Euler–Lagrange equation, as can be directly checked by using the coordinate expression for l_ω and for the adjoint operator obtained in 6.7 and

7.3. Thus the theorem proved above is an invariant global version of the classical Euler–Lagrange theorem.

Now suppose $B \subset N_m^\infty$. Consider two charts on N_m^∞ which come from the bundles π and $\tilde{\pi}$, denote by l_ω ($\mathfrak{Z}_\varphi$) and $\tilde{l}_\omega$ ($\tilde{\mathfrak{Z}}_{\tilde{\varphi}}$) the linearization operators (the evolutionary derivation operators) on the first and second of them, respectively. Suppose that for $\chi \in \kappa(N_m^\infty)$ within each of the charts we have the relations $\mathfrak{Z}_\varphi = \chi$, $\tilde{\mathfrak{Z}}_{\tilde{\varphi}} = \chi$. Then on the intersection of the charts we have $\tilde{\mathfrak{Z}}_{\tilde{\varphi}} = \mathfrak{Z}_\varphi + X$, $X \in \mathscr{C}D(A)$, $\tilde{\varphi} = C\varphi$, where $C: \mathscr{F}(\pi, \pi) \to \mathscr{F}(\tilde{\pi}, \tilde{\pi})$ is an isomorphism defined on the same intersection. If $V \in \mathscr{M}^n_{\mathrm{adm}}(N)$ and $\varphi = 0$ in a neighbourhood of $j(V)(\partial V) \subset N_m^\infty$ we have $\tilde{\varphi} = 0$ in this neighbourhood and therefore $X = 0$ in it. Since

$$\begin{aligned} \tilde{l}_\omega(\tilde{\varphi}) &= \tilde{\mathfrak{Z}}_{\tilde{\varphi}}(\omega) = \mathfrak{Z}_\varphi(\omega) + X(\omega) = l_\omega(\varphi) + \bar{d}(X \lrcorner\, \omega) \\ &= \langle \varphi, l_\omega^*(1) \rangle + \bar{d}\{\text{Ж}_\lambda(l_\omega(\varphi, 1)) + X \lrcorner\, \omega\} \end{aligned}$$

and $X \lrcorner\, \omega = 0$ near $j(V)(\partial V)$ we have

$$\begin{aligned} \int_V j(V)^* \langle \tilde{\varphi}, \tilde{l}_\omega^*(1) \rangle &= \int_V j(V)^* \tilde{l}_\omega(\tilde{\varphi}) \\ &= \int_V j(V)^* \langle \varphi, l_\omega^*(1) \rangle = \int_V \langle j(V)^* \varphi, j(V)^* l_\omega^*(1) \rangle. \end{aligned}$$

This shows that on the intersection of the charts the equations $l_\omega^*(1) = 0$ and $\tilde{l}_\omega^*(1) = 0$ coincide. In other words, this means that the Euler–Lagrange equation is invariant under substitutions of dependent and independent variables and also shows that the Euler–Lagrange equation, defined on the charts of some covering of N_m^∞ can be glued together into a unique Euler–Lagrange equation on N_m^∞ and this result does not depend on the choice of the covering.

In constructing the Euler–Lagrange equation on N_m^∞ is desirable to eliminate the gluing together operations. This will be done below (see 9.7). For now, the operator on $J^\infty(\pi)$

$$\mathscr{E}: \bar{H}^n(B, \partial B) \to \hat{\kappa}(\pi)|_B, \ \mathscr{E}(\mathscr{L}) = l_\omega^*(1),$$

where $\mathscr{L} = \langle \omega \rangle$ will be called the Euler operator.

8.4. *The "Manifolds"* ∂B

Intuitively, the manifold ∂B should be defined by setting

$$\partial B = \{ j(\sigma)(x) \in J^\infty(\pi) \mid x \in \partial \mathscr{U}, \sigma \in \Gamma_{\mathrm{adm}}(\pi), \\ \mathscr{U} \text{ is the domain of } \sigma \}$$

or

$$\partial B = \{ j(V)(x) \in N_m^\infty \mid x \in \partial V, V \in \mathscr{M}^n_{\mathrm{adm}}(N) \}.$$

However, the boundary conditions participate in the definition of the $\Gamma_{\mathrm{adm}}(\pi)$

and $\mathscr{M}^n_{\mathrm{adm}}(N)$ so that this would lead to a vicious circle. As a rule, in natural variational problems arising in geometry and physics, the boundary conditions are clearly a priori or are directly included in the statement of the problem. In these cases the sets $\Gamma_{\mathrm{adm}}(\pi)$ and $\mathscr{M}^n_{\mathrm{adm}}(N)$ are given and we can use the definition of ∂B given above. For an arbitrary Lagrangian apparently there does not exist any reliable a priori considerations concerning the type of manifold ∂B one should consider. Therefore we shall postulate one general requirement motivated in the following way.

Suppose $\mathscr{I} \subset \mathscr{F}_m(N)$ (resp. $\mathscr{I} \subset \mathscr{F}(\pi)$) is an ideal of the "manifold" ∂B. If $L \in \mathscr{M}^n_{\mathrm{adm}}(N)$ (resp. $\sigma \in \Gamma_{\mathrm{adm}}(\pi)$) then $j(L)(\partial L) \subset \partial B$ (resp. $j(\sigma)(\partial \mathscr{U}) \subset \partial B$).

In the general position situation the intersection of the tangent spaces to $\operatorname{im} j(L)$ (resp. $\operatorname{im} j(\sigma)$) and ∂B is of dimension $n-1$ at the points $\theta \in j(L)(\partial L)$ (resp. $\theta \in j(\sigma)(\partial \mathscr{U})$). This means that $j(L)^* (\mathscr{I}) \subset C^\infty(L)$ is the ideal which determines ∂L. For this reason the rank of the system of differentials $d\varphi$, $\varphi \in j(L)^* (\mathscr{I})$ equals 1. But $d\{ j(L)^* (\mathscr{I})\} = j(L)^* (\bar{d}\mathscr{I})$. It is therefore necessary that the rank of the system of differentials $\bar{d}f, f \in \mathscr{I}$ be equal to 1 or in other words that the $A/\mathscr{I}$-module $\mathscr{B} = A \cdot \bar{d}\mathscr{I} / \mathscr{I} \cdot \bar{d}\mathscr{I}$ be one dimensional.

Thus, for the submanifold ∂B we shall consider only manifolds for which the $A/\mathscr{I}$-module $\mathscr{B}$ is one dimensional. It is appropriate to call such manifolds ∂-admissible just as we call the ideals which determine them. Practically we can start from an arbitrary submanifold (ideal) and the "cut off" ("enlarge") it by adding conditions of one dimensionality.

The algebra $A' = A/\mathscr{I}$ is naturally filtered by the subalgebras $A'_m = A_m/\mathscr{I} \cap A_m$. In the FG-category over A' we can introduce the operation $\mathscr{C}$ by putting $\mathscr{C}\Phi_{A'} = \mathscr{C}\Phi_A|_{\partial B}$, where Φ_A ($\Phi_{A'}$) are the representative object for the functor Φ of the differential calculus in the FG-category over A (A'). It follows from the condition of one dimensionality of $\mathscr{B}$ that maximal integral manifolds of the "distribution $\mathscr{C}$" on ∂B (i.e., those $V \subset \partial B$ for which $\mathscr{C}\Phi_{A'}|_V = 0$, $\forall \Phi_{A'}$) are of the dimension $n-1$, except those which are contained in $\overline{\partial B} = \{\theta \in \partial B \mid \mathscr{B}_\theta = 0\}$.

Further the algebra A' (or ∂B) is viewed as an object in the category DE. The notations for the standard modules related to the differential calculus on A' remain the same except that the symbol ∂B is added whenever necessary, e.g., $\kappa(\partial B)$, $\bar{\Lambda}^i(\partial B)$, etc.

8.5. *Transversality Conditions*

Suppose the $\mathscr{C}$-field $X = Э_\chi \in Y$, $Y + \mathscr{C}D(A)$, is tangent to ∂B, where $A = \mathscr{F}(\pi)$ or we are working within an affine chart. Then

$$X(\omega) = l_\omega(\chi) + \bar{d}(Y \lrcorner\, \omega)$$
$$= \langle \chi, l^*_\omega(1)\rangle + \bar{d}\{Ж_\lambda(l_\omega(\chi, 1) + Y \lrcorner\, \omega\}$$

and the velocity of change of the functional $\mathscr{L} = \langle\omega\rangle$ along the field X on $\sigma \in \Gamma_{\mathrm{adm}}(\pi)$ under the condition that σ is a solution of the equation $\mathscr{E}\{\mathscr{L}) = 0$ is

$$\dot{\mathscr{L}}_\sigma = \int_{\mathscr{U}} d\{Ӝ_{\lambda(\sigma)}(l_{\omega,\sigma}(\chi, 1) + Y_\sigma \lrcorner\, \omega_\sigma\}$$

$$= \int_{\partial\mathscr{U}} \{Ӝ_{\lambda(\sigma)}(l_{\omega,\sigma}(\chi, 1) + Y_\sigma \lrcorner\, \omega_\sigma\},$$

where $\omega_\sigma = j(\sigma)^*(\omega)$ and $Y_\sigma = j(\sigma)^* \circ Y \circ \pi_\infty^* \in D(\mathscr{U})$. Thus a necessary condition of extremality, complementary to the equation $\mathscr{Y} = \{\mathscr{E}(\mathscr{L}) = 0\}$, is

$$0 = \dot{\mathscr{L}}_\sigma = \int_{\partial\mathscr{U}} \{Ӝ_{\lambda(\sigma)}(l_{\omega,\sigma}(\chi, 1) + Y_\sigma \lrcorner\, \omega_\sigma\}$$

$$= (j(\sigma)|_{\partial\mathscr{U}})^* \langle Ӝ_\lambda(l_\omega(\chi, 1) + Y \lrcorner\, \omega\rangle_{\partial B \cap \mathscr{Y}_\infty}$$

for all $X = Э_\chi + Y$ tangent to ∂B, where $\langle Ӝ_\lambda(l_\omega(\chi, 1) + Y \lrcorner\, \omega\rangle_{\partial B \cap \mathscr{Y}_\infty} \in \bar{H}^{n-1}(\partial B \cap \mathscr{Y}_\infty)$ is the Rham $\bar{d}$-cohomology class of the $(n-1)$-form $(Ӝ_\lambda(l_\omega(\chi, 1) + Y \lrcorner\, \omega)|_{\partial B \cap \mathscr{Y}_\infty}$. Note that $Э_\chi \lrcorner\, \omega = 0$, since $Э_\chi$ is a vertical field, while the form ω is horizontal, so that $Y \lrcorner\, \omega = X \lrcorner\, \omega$. Put $b_\lambda(\omega, X) = Ӝ_\lambda(l_\omega(\chi, 1) + X \lrcorner\, \omega$. Then the extremality condition given above may be written in the form

$$(j(\sigma)|_{\partial\mathscr{U}})^* \langle b_\lambda(\omega, X)\rangle|_{\partial B \cap \mathscr{Y}_\infty} = 0.$$

This may be called the implicit transversality condition. The integral character of this condition and its explicit dependence on X result in it being not practically useful. Further it shall be written in an explicit local form.

LEMMA. *$b_\lambda(\omega, X)|_{\partial B \cap \mathscr{Y}_\infty} = 0$ if $X|_{\partial B \cap \mathscr{Y}_\infty} = 0$.*

Proof. First, $(X \lrcorner\, \omega)|_{\partial B \cap \mathscr{Y}_\infty} = 0$ since $X|_{\partial B \cap \mathscr{Y}_\infty} = 0$. Second, $(Э_\chi)_\theta = 0$ if $X_\theta = 0$, $\theta \in J^\infty(\pi)$, so that $(Э_\chi)_\theta = 0$ if $\theta \in \partial B \cap \mathscr{Y}_\infty$. But $Э_\chi(p)_\theta = l_p(\chi)_\theta = 0$, $\forall p \in P$. Therefore for any $\nabla \in \mathscr{C}\,\mathrm{Diff}(\kappa, P)$, $\nabla(\chi)_\theta = 0$ because operators of the form l_p, $p \in P$, generate $\mathscr{C}\,\mathrm{Diff}(\kappa, P)$. But the operator

$$\nabla: \kappa \to \bar{\Lambda}^{n-1}, \qquad \nabla(\chi) = Ӝ_\lambda(l_\omega(\chi, 1)),$$

is $\mathscr{C}$-differential. This immediately follows from the fact that such is $Ӝ_\lambda$ and $l_\omega(\chi, 1) = l_\omega \circ \chi$. Therefore $Ӝ_\lambda(l_\omega(\chi, 1))|_{\partial B \cap \mathscr{Y}_\infty} = 0$. ∎

DEFINITION. The boundary ∂B will be called normal ($\mathscr{L}$-normal) if $\overline{\partial B} = \varnothing$ ($\overline{\partial B} \cap \mathscr{Y}_\infty = \varnothing$) and every $\mathscr{C}$-field on ∂B (on some neighbourhood $\partial B \cap \mathscr{Y}_\infty$ in ∂B) can be extended to a $\mathscr{C}$-field on B (to some neighbourhoods

$\mathscr{Y}_\infty$ in B). It is obvious that $\mathscr{L}$-normality is a consequence of normality. If the boundary ∂B is $\mathscr{L}$-normal then, in view of the lemma proved above the operator

$$b_{\omega,\lambda} \in \operatorname{Diff}(D_{\mathscr{C}}(\partial B)|_{\partial B \cap \mathscr{Y}_\infty}, \bar{\Lambda}^{n-1}(\partial B \cap \mathscr{Y}_\infty))$$

is well defined by the relations

$$b_{\omega,\lambda}(Z) = b_\lambda(\omega, X)|_{\partial B \cap \mathscr{Y}_\infty},$$

where X is a $\mathscr{C}$-field in some neighbourhood of $\mathscr{Y}_\infty$ in B tangent to ∂B and $X|_{\mathscr{U}} = Z$, where $\mathscr{U} \subset \partial B$ is the domain of Z.

PROPOSITION. *The operator $b_{\omega,\lambda}$ possesses the following properties*:

(1) $b_{\omega,\lambda} = b_{\omega',\lambda}$ *if* $\langle\omega\rangle = \langle\omega'\rangle \in \bar{H}^n(B, \partial B)$,

(2) $b_{\omega,\lambda}|_{\mathscr{C}D(\partial B, \mathscr{Y}_\infty)} = 0$, *where* $\mathscr{C}D(\partial B, \mathscr{Y}_\infty) = \mathscr{C}D(\partial B)|_{\mathscr{Y}_\infty}$.

Proof. (1) Since X is tangent to ∂B we have

$$X(\bar{d}v)|_{\mathscr{U}} = \bar{d}(X(v)|_{\mathscr{U}}) = \bar{d}(Z(v|_{\mathscr{U}})).$$

Therefore $X(\bar{d}v)|_{\mathscr{U}} = 0$ if $v|_{\mathscr{U}} = 0$. But $\omega' = \bar{d}v + \omega$, $v|_{\partial B} = 0$ so that $b_{\omega',\lambda} - b_{\omega,\lambda} = b_{\bar{d}v,\lambda}$ while on the other hand $b_{\bar{d}v,\lambda}(Z) = X(\bar{d}v)|_{\partial B \cap \mathscr{Y}_\infty} = 0$.

(2) Suppose $\tilde{X} \in \mathscr{C}D(W)$, where W is some neighbourhood of $\partial B \cap \mathscr{Y}_\infty$ in ∂B while X is the $\mathscr{C}$-field prolongating $\tilde{X}$ in some neighbourhood in $\mathscr{Y}_\infty$. If $\sigma \in \Gamma(\pi)$ is an extremal and $V \subset M$ is its domain, then the field $\tilde{X}$ and therefore X are tangent to $j(\sigma)(\partial V) \subset \partial B \cap \mathscr{Y}_\infty$. Therefore, in particular, if $\theta \in j(\sigma)(\partial V)$ and $X = \mathfrak{Z}_f + Y$, $Y \in \mathscr{C}D$, then $\mathfrak{Z}_f|_\theta = 0$. The argument applied above to prove the lemma shows that $\mathfrak{K}_\lambda(l_\omega(\chi, 1))|_\theta = 0$. Further $j(\sigma)^* (X \lrcorner\, \omega) = X_\sigma \lrcorner\, \omega_\sigma$, where $\omega_\sigma = j(\sigma)^* \omega$ and $X_\sigma = j(\sigma)^{-1}(X'_\sigma)$, where X'_σ is the restriction of X to $j(\sigma)(V)$. Since X_σ is tangent to ∂V we have $0 = (X_\sigma \lrcorner\, \omega_\sigma)|_{\partial V} = j(\sigma)^* (X)|_{\partial V}$, i.e., $(X \lrcorner\, \omega)|_{\partial B \cap \mathscr{Y}_\infty} = 0$. ∎

The submanifolds $G = \partial B$ or $= \partial B \cap \mathscr{Y}_\infty$ in N^∞_m (or $J^\infty(\pi)$) may be viewed as objects of DE. Indeed, if $A = \mathscr{F}_n(N)$ (or $= \mathscr{F}(\pi)$), then the subalgebras $\mathscr{F}_i(G) = A_i|_G$ determine a filtration in $\mathscr{F}(G) = A|_G$. Moreover if $\Phi(G)$ is the representative object for a functor of the differential calculus Φ in the FG-category over $\mathscr{F}(G)$ while Φ is one over A, then we put $\mathscr{C}\Phi(G) = \mathscr{C}\Phi|_G$.

Statement (2) shows that the operator $b_{\omega,\lambda}$ generates a quotient operator

$$\Gamma_{\omega,\lambda}\colon \kappa(\partial B, \mathscr{Y}_\infty) \to \bar{\Lambda}^{n-1}(\partial B \cap \mathscr{Y}_\infty),$$

where

$$\kappa(\partial B, \mathscr{Y}_\infty) = \kappa(\partial B)|_{\mathscr{Y}_\infty} = D_{\mathscr{C}}(\partial B)|_{\partial B \cap \mathscr{Y}_\infty} / \mathscr{C}D(\partial B)|_{\partial B \cap \mathscr{Y}_\infty}.$$

COROLLARY. *The operator* $\Gamma_{\omega,\lambda}$ *is* $\mathscr{C}$*-differential over* $\partial B \cap \mathscr{Y}_\infty$ *and* $\Gamma_{\omega,\lambda'} = \Gamma_{\omega,\lambda} + \bar{d} \circ \nabla$, $\nabla \in \mathscr{C}$ Diff.

Proof. Indeed, as was pointed out in the previous subsection, the maximal integral submanifold in ∂B are of the form $j(\sigma)(\partial V)$, where V is the domain of σ. In our proof of statement (2) we actually showed that if $(j(\sigma)|_{\partial V})^*\,(h) = 0$, $h \in \kappa(\partial B, \mathscr{Y}_\infty)$, then $b_{\omega,\lambda}(Z) = 0$, where Z is the $\mathscr{C}$-field which determines h. This means that the operator $\Gamma_{\omega,\lambda}$ possesses a restriction to any maximal integral submanifold in $\partial B \cap \mathscr{Y}_\infty$, i.e., it is $\mathscr{C}$-differential.

Since $\mathcal{K}_{\lambda'} \circ l_\omega = \mathcal{K}_\lambda \circ l_\omega + \bar{d} \circ \Box$, $\Box \in \mathscr{C}\,\mathrm{Diff}(\kappa, \Lambda^{n-2})$ we have $\Gamma_{\omega,\lambda'}(h) = \Gamma_{\omega,\lambda}(h) + \bar{d}\nabla(h)$, where $\nabla(h) = \Box(Y)|_{\partial B \cap \mathscr{Y}_\infty}$ and $Y \subset D_{\mathscr{C}}(B)$ is tangent to ∂B near $\mathscr{Y}_\infty$ while $h = Y|_{\partial B \cap \mathscr{Y}_\infty} \bmod \mathscr{C}D(\partial B)|_{\partial B \cap \mathscr{Y}_\infty}$. The fact that the operator ∇ is well defined in this manner is shown just as statement (2) of Proposition 8.5, while the fact that it is $\mathscr{C}$-differential is shown similarly to the same property of the operator $\Gamma_{\omega,\lambda}$. ∎

Taking into consideration the definition of the operator $\Gamma_{\omega,\lambda}$, we can rewrite the implicit transversality condition in the following way: $(j(\sigma)|_{\partial V})^*\,\langle \Gamma_{\omega,\lambda}(h)\rangle = 0$, where $h \in \kappa(\partial B, \mathscr{Y}_\infty)$ corresponds to $X|_{\partial B \cap \mathscr{Y}_\infty}$.

Now assume that the module $\Lambda^{n-1}(\partial B \cap \mathscr{Y}_\infty)$ is locally free and one dimensional. In the $\mathscr{C}$-theory over $\partial B \cap \mathscr{Y}_\infty$ the adjoint operator theory can be constructed along the same lines as in the previous section. Then, applying Green's $\mathscr{C}$-formula to the operator $\Gamma_{\omega,\lambda}$ we find that

$$\langle \Gamma_{\omega,\lambda}(h)\rangle = \langle (h, \Gamma^*_{\omega,\lambda}(1)\rangle \in \bar{H}^{n-1}(\partial B \cap \mathscr{Y}_\infty).$$

It now follows from the Dubois–Raymond lemma that the implicit transversality condition will hold if and only if

$$(j(\sigma)|_{\partial V})^*\,(\Gamma^*_{\omega,\lambda}(1)) = 0.$$

It now follows from statements (1) and (2) of the proposition proved above and the fact that $\bar{d}^* = \pm\bar{d}$ that $\Gamma^*_{\omega,\lambda}(1)$ does not depend on the choice of λ and of the form ω in the class $\mathscr{L} = \langle\omega\rangle$. Thus, from the Lagrangian $\mathscr{L}$ we can construct an unique fashion the element $\Gamma(\mathscr{L}) = \Gamma^*_{\omega,\lambda}(1) \in \hat{\kappa}(\partial B, \mathscr{Y}_\infty)$. And in this case the condition

$$(j(\sigma)|_{\partial V})^*\,(\Gamma(\mathscr{L})) = 0$$

is a necessary condition for the extremality of the section σ. Further we will call this the transversality condition since it exactly corresponds to the classical transversality condition in standard problems of the calculus of variations with a free boundary.

Now assume that the Lagrangian is of the form $\mathscr{L} = \mathscr{L}' + \mathscr{L}''$, where $\mathscr{L}' \in \bar{H}^n(B, \partial B)$, $\mathscr{L}'' \in \bar{H}^{n-1}(\partial B)$. If the $\mathscr{C}$-field X is tangent to ∂B and

$X_1 = X|_{\partial B}$, then $X(\mathscr{L}) = X(\mathscr{L}') + X_1(\mathscr{L}'')$. In particular, the consideration of fields X such that $X_1 = 0$ shows that the Lagrange–Euler equation for the Lagrangian $\mathscr{L}'$ is a necessary condition for the extremum of a functional corresponding to the Lagrangian $\mathscr{L}$. Therefore

$$X(\mathscr{L})|_{\mathscr{Y}_\infty} = \bar{d}\{\text{Ж}_\lambda(l_\omega(f, 1)) + X \lrcorner\, \omega\}|_{\mathscr{Y}_\infty} + X_1(\omega_1)|_{\partial B \cap \mathscr{Y}_\infty},$$

where $\mathscr{L}' = \langle \omega \rangle$, $\mathscr{L}'' = \langle \omega_1 \rangle$, $X = \text{Э}_f + Y$, and the velocity of the change of $\mathscr{L}_\sigma$ (= the functional under consideration on the section σ) along the field X equals

$$\begin{aligned} \dot{\mathscr{L}}_\sigma &= (j(\sigma)|_{\partial V})^* \langle \text{Ж}_\lambda(l_\omega(f, 1)) + X \lrcorner\, \omega + X(\omega_1)\rangle|_{\partial B \cap \mathscr{Y}_\infty} \\ &= (j(\sigma)|_{\partial V})^* \langle \Gamma_{\omega,\lambda}(h) + X_1(\omega_1)\rangle|_{\partial B \cap \mathscr{Y}_\infty} \in \bar{H}^{n-1}(\partial B \cap \mathscr{Y}_\infty), \quad (8.5.1) \end{aligned}$$

where $h \in \kappa(\partial B, \mathscr{Y}_\infty)$ corresponds to X_1.

If $\partial B \cap \mathscr{Y}_\infty$, as an object of DE, is situated regularly in the DE-objects ∂B in the sense that for every point $y \in \partial B \cap \mathscr{Y}_\infty$ there is a neighbourhood $W \subset \partial B$ which, as an object of DE, locally has the form $J^\infty(\pi_1)$, where π_1 is some bundle (possibly infinite dimensional) over a $(n-1)$-dimensional base, then in formula (8.5.1) we can get rid of X. To do this, identify W and $J^\infty(\pi_1)$. Then $X_1|_W = \text{Э}_{f_1} + Y$, $Y \in \mathscr{C}D(W)$ and if $h = f_1|_{\partial B \cap \mathscr{Y}_\infty}$ then

$$\begin{aligned} X_1(\omega_1) &= l_{\omega_1}(f_1) + \bar{d}(X_1 \lrcorner\, \omega_1) \\ &= (l^*_{\omega_1}(1), h) + \bar{d}\{\text{Ж}_{\lambda_1}(l_{\omega_1}(h, 1) + X_1 \lrcorner\, \omega_1\}. \end{aligned}$$

Therefore $\dot{\mathscr{L}}_\sigma = (j(\sigma)|_{\partial V})^* \langle \Gamma(\mathscr{L}') + (\tilde{l}^*_{\omega_1}(1), h)\rangle$, where $\tilde{l}_{\omega_1} = l_{\omega_1}|_{\mathscr{Y}_\infty \cap W}$ and by the Dubois–Raymond lemma we obtain the following extremality condition for the section σ:

$$(j(\sigma)|_{\partial V})^* \mathscr{E}^{(1)}(\mathscr{L}) = 0.$$

where $\mathscr{E}^{(1)}(\mathscr{L}) = \Gamma(\mathscr{L}') + \tilde{l}^*_{\omega_1}(1) = \Gamma(\mathscr{L}') + \mathscr{E}(\mathscr{L}'')$.

8.6. *Conservation Laws*

Suppose $B \subset J^\infty(\pi)$, $\mathscr{L} = \langle \omega \rangle$, $\omega \in \Lambda^n_0$. Then the field $X \in D_{\mathscr{C}}(B)$ will be called an (infinitesimal) symmetry of the density of the Lagrangian ω, if $X(\omega) = 0$ on $\mathscr{Y}_\infty$, where $\mathscr{Y} = \{\mathscr{E}(\mathscr{L}) = 0\}$. As we saw above on $\mathscr{Y}_\infty$ we have

$$X(\omega) = \bar{d}\{\text{Ж}_\lambda(l_\omega \circ \chi) + X \lrcorner\, \omega\} = \bar{d}b_\lambda(\omega, X),$$

where $X = \text{Э}_\chi + Y$, $Y \in \mathscr{C}D(B)$. Putting $c_\lambda(\omega, X) = b_\lambda(\omega, X)|_{\mathscr{Y}_\infty}$ we see that $\bar{d}c_\lambda(\omega, X) = 0$, i.e., $\langle c_\lambda(\omega, X)\rangle \in \bar{H}^{n-1}(\mathscr{Y}_\infty)$.

Definition. The cohomology class $H \in \bar{H}^{n-1}(\mathscr{Y}_\infty)$ will be called a

(local) conservation law for the equaton $\mathscr{Y}$. If $H = \langle\rho\rangle$, $\rho \in \bar{\Lambda}^{n-1}(\mathscr{Y}_\infty)$ then ρ will be called a c-density.

It can easily be seen from this definition that for any solution $V \subset N$ (or $\sigma \in \Gamma_{\text{loc}}(\pi)$) of the equation $\mathscr{Y}$ the form $j(V)^*(\rho)$ (or $j(\sigma)^*(\rho)$) is closed on V (on $\mathscr{U}_\sigma$) and its cohomology class is $j(V)^*(H)$ ($j(\sigma)^*(H)$). Therefore the expression $(j(V)^*(H), [W^{n-1}]) = \int_W j(V)^*(\rho)$ (resp. $(j(\sigma)^*(H), [W^{n-1}]) = \int_W j(\sigma)^*(\rho)$) for any oriented submanifold $W^{n-1} \subset V$ (or $W^{n-1} \subset \mathscr{U}_\sigma$) is an invariant expression in the sense that it does not change when W^{n-1} is replaced by any manifold which is homological to it. If one of the independent variables of the equation $\mathscr{Y}$ is the time t, then the surfaces $\{t = t_1\}$ and $\{t = t_2\}$ are homological to each other and the integrals written above over them coincide, if they are defined. This motivates the definition given above from the "physical" point of view.

It follows from what we said above immediately that a result which should properly be called the generalized Noether theorem, holds.

THEOREM. *If X is a symmetry of the density of the Lagrangian ω, then $c_\lambda(\omega, X)$ is the c-density for the corresponding Euler–Lagrange equation while $\langle c_\lambda(\omega, X)\rangle$ is its conservation law which does not depend on the choice of λ.*

If $X(\omega)|_{\mathscr{Y}_\infty} = \bar{d}v$ then obviously the form $c_\lambda(\omega, X) - v \in \bar{\Lambda}^{n-1}(B)$ is a c-density for the Euler–Lagrange equation $\mathscr{Y}$. This fact is often useful in the practical search for conservation laws. However, it reduces to the previous theorem.

Indeed, if $Y \in \mathscr{C}D(B)$ is such that $(Y \lrcorner\, \omega)|_{\mathscr{Y}_\infty} = v$, then $Y(\omega)|_{\mathscr{Y}_\infty} = \bar{d}v$ and $c_\lambda(\omega, X - Y) = c_\lambda(\omega, X) - v$.

The (infinitesimal) symmetry transformations which appears in Noether's classical theorem, is generated by the (infinitesimal) transformation of dependent and independent variables, i.e., by the vector field Y^0 on $J^0(\pi)$. If it is naturally lifted to $J^k(\pi)$ in the case $\omega \in \Lambda_0^n(J^k(\pi))$ it generates a transformation of the density of the Lagrangian ω. This lifting, which we denote by $Y^{(k)}$, $0 \leqslant k \leqslant \infty$, may be described as follows. First of all let $f = Y \lrcorner\, U_1(\pi) \in \mathscr{F}_1(\pi, \pi)$ and $X_f = \mathfrak{Z}_f - l_f$ (see [13]). Then the operator $Y^{(\infty)} = Y_f$ is uniquely determined by the relation

$$X_f(\varphi\chi) = Y_f(\varphi)\chi + \varphi X_f(\chi), \qquad \varphi \in \mathscr{F}(\pi),\ \chi \in \mathscr{F}(\pi, \pi) \approx \kappa(\pi).$$

The operator Y_f preserves the filtration $\{\mathscr{F}_k(\pi)\}$ and therefore its restriction $Y^{(k)}$ to $\mathscr{F}_k(\pi)$ is defined (see [4, 13]). The assumption $Y^{(k)}(\omega) = 0$ of the classical Noether theorem is obviously equivalent to $Y_f(\omega) = 0$ and the conservation law, described by the theorem stated above for the field Y_f coincides with the one given by the Noether theorem. Let us stress that

$Y_f = 3_f + Z$, $Z \in \mathscr{C}D(\mathscr{F}(\pi))$. Thus the theorem that we have proved admits, if compared to the classical Noether theorem, a much larger class of transformation since the fields which appear in it are of the form $X = 3_f + Z$, where $f \in \mathscr{F}(\pi, \pi)$ but not only $\mathscr{F}_1(\pi, \pi)$.

Recall that if $\chi \in \operatorname{Sym} \mathscr{Y}$, while $H \in \bar{H}^i(\mathscr{Y}_\infty)$ then the Lie derivative $\chi(H) = \langle X(\omega) \rangle \in \bar{H}^i(\mathscr{Y}_\infty)$ is well defined, where

$$H = \langle \omega \rangle, \qquad X \in D_{\mathscr{C}}(\mathscr{Y}_\infty), \qquad \omega \in \bar{\Lambda}^i(\mathscr{Y}_\infty),$$
$$\chi = X \bmod \mathscr{C}D(\mathscr{Y}_\infty),$$

since $Y(\omega) = \bar{d}(Y \rfloor \omega)$, if $Y \in \mathscr{C}D(\mathscr{Y}_\infty)$ and $X \circ \bar{d} = \bar{d} \circ X$. For this reason we have the following mechanism for getting new conservation laws from old ones.

PROPOSITION. *If χ is a symmetry of the equation $\mathscr{Y}$ while H is its conservation law, then $\chi(H)$ is also its conservation law.*

9. The $\mathscr{C}$-Spectral Sequence

In this section we construct the $\mathscr{C}$-spectral sequence $\{E_r^{p,q}, d_r^{p,q}\}$, establish some of its properties and give the first applications. In particular, it is shown that the Euler operator coincides with the differential $d_1^{0,n}$ of the spectral sequence which yields a solution "in the large" of the Lagrangian triviality problem and the inverse problem of the calculus of variations in the situation when non-holonomic constraints are absent. Moreover, we obtain the "infinitesimal Stokes formula" for the term E_1 which is the basis of homotopy methods for computing E_2 and, among other things, enables us to indicate explicit formulas in the inverse problem of the calculus of variation and the Lagrangian triviality problem. The $\mathscr{C}$-spectral sequence is natural in the category DE. This in particular proves the permutability of the Euler operator with morphisms of DE which considerably generalizes the well-known invariance property of the Euler–Lagrange equations under changes of dependent and independent variables.

9.1. *The $\mathscr{C}$ Spectral Sequence*

Suppose $\mathscr{Y}$ is some object of the category DE (see 6.4), $\Lambda^i = \Lambda^i(\mathscr{Y})$, $\Lambda^* = \Lambda^*(\mathscr{Y})$. It is obvious that $\mathscr{C}\Lambda^* = \mathscr{C}\Lambda^*(\mathscr{Y})$ is an ideal of the exterior algebra Λ^* and it is stable with respect to the operator d. For this reason all its powers $\mathscr{C}^k\Lambda^* = \mathscr{C}^k\Lambda^*(\mathscr{Y})$ have the same property. Thus, in the de Rham complex $\{\Lambda^*, d\}$ on $\mathscr{Y}$ we obtain the filtration

$$\Lambda^* = \mathscr{C}^0\Lambda^* \supset \mathscr{C}^1\Lambda^* = \mathscr{C}\Lambda^* \supset \cdots \supset \mathscr{C}^k\Lambda \supset \cdots.$$

Consider the spectral sequence $\{E_r^{p,q}, d_r^{p,q}\} = \{E_r^{p,q}(\mathscr{Y}), d_r^{p,q}(\mathscr{Y})\}$ determined by this filtration and called further the $\mathscr{C}$-spectral sequence for $\mathscr{Y}$. As usual p is the filtration index, and $p+q$ is the degree. Multiplication in Λ^* induces in the usual way a multiplication in the terms E_r and we have $E_r^{p,q} \wedge E_r^{s,t} \subset E_r^{p+s,q+t}$ while

$$d_r(\omega_1 \wedge \omega_2) = d_r\omega_1 \wedge \omega_2 + (-1)^{p+q}\,\omega_1 \wedge d_r\omega_2, \qquad \omega_1 \in E_r^{p,q},\ \omega_2 \in E_r^{s,t}.$$

PROPOSITION. (1) *If $F\colon \mathscr{Y}_1 \to \mathscr{Y}_2$ is a morphism in* DE *then F^* generates the homomorphism of the $\mathscr{C}$-spectral sequence for $\mathscr{Y}_2$ into the $\mathscr{C}$-spectral sequence for $\mathscr{Y}_1$.*

(2) $E_0^{0,q} = \bar{\Lambda}^q$, $E_1^{0,q} = \bar{H}^q$.

(3) $E_0^{p,q} = 0$ *if* $q < 0$.

(4) *If for some $s \geqslant 0$ we have $\bar{\Lambda}^{s+1} = 0$ then $E_0^{p,q} = 0$ for $q > s$.*

(5) *The $\mathscr{C}$-spectral sequence converges and its term $E_\infty = \{E_\infty^{p,q}\}$ is attached to $H^*(\mathscr{Y})$.*

Proof. (1) and (2) immediately follow from definitions. Further $\mathscr{C}\Lambda^i = \mathscr{C}\Lambda^1 \wedge \Lambda^{i-1}$. Therefore $\mathscr{C}^k\Lambda^i = 0$, if $k > i$. This proves (3). If $\bar{\Lambda}^{s+1} = 0$ then $\mathscr{C}\Lambda^{s+1} = \Lambda^{s+1}$ and generally $\mathscr{C}\Lambda^i = \Lambda^i$ whenever $i > s$ since $\mathscr{C}\Lambda^{s+1} \wedge \Lambda^{i-s-1} \subset \mathscr{C}\Lambda^i$ while on the other hand $\mathscr{C}\Lambda^{s+1} \wedge \Lambda^{i-s-1} = \Lambda^{s+1} \wedge \Lambda^{i-s-1} = \Lambda^i$. Also it follows from the relation $\mathscr{C}\Lambda^{m+1} = \mathscr{C}\Lambda^1 \wedge \Lambda^m$ that

$$\mathscr{C}^k\Lambda^{m+k} = \underbrace{\mathscr{C}\Lambda^1 \wedge \cdots \wedge \mathscr{C}\Lambda^1}\wedge \Lambda^m \ (k \text{ times}).$$

Therefore when $q > s$ we have

$$\begin{aligned}\mathscr{C}^p\Lambda^{p+q} &= \underbrace{\mathscr{C}\Lambda^1 \wedge \cdots \wedge \mathscr{C}\Lambda^1}_{p\text{ times}} \wedge \Lambda^q \\ &= \underbrace{\mathscr{C}\Lambda^1 \wedge \cdots \wedge \mathscr{C}\Lambda^1}_{p\text{ times}} \wedge \mathscr{C}\Lambda^q \subset \mathscr{C}^{p+1}\Lambda^{p+q} \subset \mathscr{C}^p\Lambda^{p+q},\end{aligned}$$

i.e., $\mathscr{C}^p\Lambda^{p+q} = \mathscr{C}^{p+1}\Lambda^{p+q}$. Finally (5) follows from the fact that, according to (3), the spectral sequence under consideration is entirely situated in the first "quadrant." ∎

COROLLARY. *If $\mathscr{Y} \subset N_m^\infty$ or $J^\infty(\pi)$ then $E_0^{p,q}(\mathscr{Y}) = 0$ whenever $q < 0$ or $q > n$.*

Proof. Indeed, in this case the number s in part (4) of the previous proposition may be set equal to $n = \dim N - m$ or $\dim M$, where M is the base of the bundle π. ∎

9.2. *The Double Complex $\Lambda^*(J^\infty(\pi))$*

To obtain the necessary description of the terms $E_0^{p,q}$ it is useful to introduce the structure of the double complex in $\Lambda^* = \Lambda^*(J^\infty(\pi))$. This may be done by using the operations $\urcorner$ (see 6.5) which yields a direct sum decomposition $\Lambda^1 = \mathscr{C}\Lambda^1 \oplus \Lambda_0^1$ and therefore the direct sum decomposition

$$\Lambda^* = \sum_{p,q \geqslant 0} \Lambda^{p,q},$$

$$\Lambda^{p,q} = \underbrace{\mathscr{C}\Lambda^1 \wedge \cdots \wedge \mathscr{C}\Lambda^1}_{p\text{ times}} \wedge \underbrace{\Lambda_0^1 \wedge \cdots \wedge \Lambda_0^1}_{q\text{ times}} = \underbrace{\mathscr{C}\Lambda^1 \wedge \cdots \wedge \mathscr{C}\Lambda^1}_{p\text{ times}} \wedge \Lambda_0^q.$$

Since $\mathscr{C}^p\Lambda^* = \mathscr{C}\Lambda^1 \wedge \cdots \wedge \mathscr{C}\Lambda^1 \wedge \Lambda^*$ we see that $E_0^{p,q}$ can be identified with $\Lambda^{p,q}$. To describe the operator d in these terms we shall need the following:

LEMMA. $d(\hat{d}\varphi) \in \Lambda_0^1 \wedge \mathscr{C}\Lambda^1$, $\varphi \in A$.

Proof. $\Im_\chi \lrcorner d(\hat{d}\varphi) = \Im_\chi(\hat{d}\varphi) - d(\Im_\chi \lrcorner \hat{d}\varphi) = \Im_\chi(\hat{d}\varphi) = \hat{d}\Im_\chi(\varphi) \in \Lambda_0^1$, $\Im_\chi \lrcorner \hat{d}\varphi = 0$ since $\Im_\chi$ is a vertical field while $\hat{d}\varphi$ is a horizontal form. Therefore $d(\hat{d}\varphi) \in \mathscr{C}\Lambda^1 \wedge \Lambda_0^1 \oplus \Lambda_0^1 \wedge \Lambda_0^1$. But for $u \in \Gamma_{\text{loc}}(\pi)$

$$j(u)^*\,(d(\hat{d}\varphi)) = d[\,j(u)^*\,(\hat{d}\varphi)] = d^2[\,j(u)^*\,(\varphi)] = 0$$

so that

$$d(\hat{d}\varphi) \in \mathscr{C}\Lambda^2 = \mathscr{C}\Lambda^1 \wedge \Lambda_0^1 \oplus \mathscr{C}\Lambda^1 \wedge \mathscr{C}\Lambda^1. \quad \blacksquare$$

COROLLARY. $d\Lambda^{p,q} \subset \Lambda^{p+1,q} \oplus \Lambda^{p,q+1}$.

Proof. First, $dU_1(\varphi) = -d\hat{d}\varphi \in \Lambda^{1,1}$ and obviously $d\Lambda^{1,0} \subset \Lambda^{1,1} \oplus \Lambda^{0,2}$. From the last inclusion, in view of the relation $\Lambda_0^q = \Lambda_0^1 \wedge \cdots \wedge \Lambda_0^1$ (q times) it follows that $d\Lambda_0^q \subset \Lambda^{0,q+1} \oplus \Lambda^{1,q}$. On the other hand, elements of the form $\rho = U_1(\varphi_1) \wedge \cdots \wedge U_1(\varphi) \wedge \omega$ where $\varphi_i \in A$, $\omega \in \Lambda_0^q$, are additive generators of $\Lambda^{p,q}$ while it follows from the above that $d\rho \in \Lambda^{p+1,q} \oplus \Lambda^{p,q+1}$. $\blacksquare$

The composition $\Lambda^{p,q} \to^d \Lambda^{p+1,q} \oplus \Lambda^{p,q+1} \to \Lambda^{p,q+1}$, where the second arrow denotes the natural projection will be denoted by $d' = d'_{p,q}$ and we shall put $d''_{p,q} = d - d' : \Lambda^{p,q} \to \Lambda^{p+1,q}$. Then under the identification of $E_0^{p,q}$ with $\Lambda^{p,q}$ described above the operator $d_0^{p,q}$ is identified with $d'_{p,q}$. Let us also note that the following formulas which are obtained immediately from a computation of the bigradings

$$d'(U_1(\varphi_1) \wedge \cdots \wedge U_1(\varphi_p) \wedge \omega_q)$$
$$= \sum_{i=1}^{p} (-1)^{i-1} U_1(\varphi_1) \wedge \cdots \wedge dU_1(\varphi_i) \wedge \cdots \wedge U_1(\varphi_p) \wedge \omega_q$$
$$+ (-1)^p U_1(\varphi_1) \wedge \cdots \wedge U_1(\varphi_p) \wedge \hat{d}\omega_q, \ \varphi_i \in A, \ \omega_q \in \Lambda_0^q, \quad (9.2.1)$$
$$d''(U_1(\varphi_1) \wedge \cdots \wedge U_1(\varphi_p) \wedge \omega_q)$$
$$= (-1)^p U_1(\varphi_1) \wedge \cdots \wedge U_1(\varphi_p) \wedge U_1(\omega_q).$$

Remark 1. The results of this subsection and their proofs remain valid for $\mathscr{Y}_\infty \subset J^\infty(\pi)$ since the operators d, $\hat{d}$, U_1 and $\urcorner$, as well as the $\mathscr{C}$-spectral sequence commute with the restriction operation to $\mathscr{Y}_\infty$.

Remark 2. Consider the spectral sequence of the de Rham cohomology of the bundle $\pi_\infty: \mathscr{Y}_\infty \to M$. Then the complex $\{\Lambda^*, d''\}$ can be naturally identified with the term E_0 of this spectral sequence.

9.3. *The Terms $E_0^{p,q}$ in the "Absolute" Case*

Let us describe the terms $E_0^{p,q}$ for an open $\mathscr{Y} \subset J^\infty(\pi)$ or N_m^∞. To do this we shall need the following:

LEMMA. *In the open domain $\mathscr{Y}$ in $J^\infty(\pi)$ or N_m^∞ let us assign to every form $\omega \in \mathscr{C}\Lambda^1$ the operator $\nabla_\omega \in \mathscr{C}$ Diff(κ, A), $\nabla_\omega(\chi) = \chi \rfloor \omega$, $\chi \in \kappa$. This correspondence is an isomorphism of the A-modules $\mathscr{C}\Lambda^1$ and $\mathscr{C}$ Diff(κ, A).*

Proof. First, $\nabla_{a\omega} = a\nabla_\omega$, $a \in A$, i.e., the operation $\omega \mapsto \nabla_\omega$ is an A-module homomorphism. If $\omega = U_1(a)$ then $\nabla_\omega(\chi) = \chi \rfloor U_1(a) = l_a(\chi)$ (see 6.6), i.e., $\nabla_\omega = l_a$. Thus, if $\omega = \sum b_i U_1(a_i)$ then $\nabla_\omega = \sum b_i l_{a_i}$. Since $\mathscr{C}$ Diff(κ, P), viewed as a A-module is generated by operators of the form l_p, $p \in P$, while the form $U_1(a)$, $a \in A$, generate $\mathscr{C}\Lambda^1$ we see that the map under consideration is an epimorphism. If $0 = \nabla_\omega = \sum b_i l_{a_i}$ then $\sum b_i U_i(a)$ also equals zero. Indeed, in this case $\nabla_\omega(\chi) = \chi \rfloor \sum b_i U_i(a) = 0$, $\forall \chi \in \kappa$. But since any vertical tangent vector at the point $\theta \in J^\infty(\pi)$ may be realized in the form $\mathfrak{Z}_\chi|_\theta$ for some χ (this can be seen, e.g., from the coordinate expression (see 6.7)), the latter means that the form $\sum b_i U_i(a)$ is horizontal and therefore equals zero since $\mathscr{C}\Lambda^1 \cap \Lambda_0^1 = 0$ (see 6.5).

Thus the lemma is proved for domains in $J^\infty(\pi)$. In view of the fact that N_m^∞ is covered by charts of the form $J^\infty(\pi)$ it is valid in this case as well. ∎

Suppose $\rho \in E_0^{p,q}$, $\chi \in \kappa$. Then since $X \rfloor \mathscr{C}^p\Lambda^* \subset \mathscr{C}^{p-1}\Lambda^*$ whenever $X \in D_{\mathscr{C}}$, while $X \rfloor \mathscr{C}^p\Lambda^* \subset \mathscr{C}^p\Lambda^*$ if $X \in \mathscr{C}D$ the substitution operation $\chi \rfloor \rho \in E_0^{0,p-1}$ is well defined as:

$$\chi \rfloor \rho = (X \rfloor \omega) \bmod \mathscr{C}^p\Lambda^{p+q-1}, \quad \chi = X \bmod \mathscr{C}D, \quad \rho = \omega \bmod \mathscr{C}^{p+1}\Lambda^{p+q}.$$

For $\rho \in E_0^{p,q}$ let us define the polydifferential operators $\nabla_\rho \in \mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{*,p}(\kappa, \bar{\Lambda}^q)$ by putting

$$\nabla_\rho(\chi_1,\dots,\chi_p) = \chi_p \rfloor (\cdots (\chi_1 \rfloor \rho) \cdots) \in \bar{\Lambda}^q, \qquad \chi_i \in \kappa. \quad (9.3.1)$$

The description of the terms $E_0^{p,q}$ are given by the following:

PROPOSITION. *For an open domain $\mathscr{Y}$ in $J^\infty(\pi)$ or N_m^∞ the operation $\rho \mapsto \nabla_\rho$ establishes an isomorphism of te A-modules $E_0^{p,q}$ and $\mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{*,p}(\kappa, \bar{\Lambda}^q)$.*

Proof. First consider the case $\mathscr{Y} \subset J^\infty(\pi)$, $p = 1$. Note that $E_0^{1,q} = \Lambda^{1,q} = \mathscr{C}\Lambda^1 \otimes \bar{\Lambda}^q$,

$$\begin{aligned}\mathscr{C}\,&\mathrm{Diff}^{\mathrm{alt}}_{*,1}(\kappa, \bar{\Lambda}^q)\\ &= \mathscr{C}\,\mathrm{Diff}_*(\kappa, \bar{\Lambda}^q) = \mathscr{C}\,\mathrm{Diff}_*(\kappa, A) \otimes \bar{\Lambda}^q\end{aligned}$$

and

$$\nabla_{\lambda \otimes \omega} = \nabla_\lambda \otimes \omega, \qquad \lambda \in \mathscr{C}\Lambda^1,\ \omega \in \bar{\Lambda}^q.$$

In view of these facts in our situation the statement we need follows from the lemma given above. From the fact that N_m^∞ is covered by charts of the form $J^\infty(\pi)$ we also obtain its validity for $p = 1$, $\mathscr{Y} = N_m^\infty$.

In order to obtain the proof in the general case let us introduce the bigraded associative algebras

$$\mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}\{P\} = \sum_{p,q} \mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{*,p}(P, \bar{\Lambda}^q) \qquad (\mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{*,0}(P, \bar{\Lambda}^q) = \bar{\Lambda}^q)$$

in which the multiplication is defined according to the rule

$$\begin{aligned}&(\Delta_1 \wedge \Delta_2)(p_1,\dots,p_{l+k})\\ &\quad = \frac{(-1)^{qk}}{l!\,k!} \sum_\sigma (-1)^\sigma \Delta_1(p_{\sigma(1)},\dots,p_{\sigma(l)}) \wedge \Delta_2(p_{\sigma(l+1)},\dots,p_{\sigma(l+k)}) \in \bar{\Lambda}^{q+k},\end{aligned}$$

where $\Delta_1 \in \mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{*,l}(P, \bar{\Lambda}^q)$, $\Delta_2 \in \mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{*,k}(P, \Lambda^r)$. Then the operation $\rho \mapsto \nabla_\rho$ determines, obviously, a homomorphism of the algebra $E_0 = \sum E_0^{p,q}$ into the algebra $\mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}\{P\}$. Also,

$$\mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{*,p}(P, \bar{\Lambda}^q) \wedge \mathscr{C}\,\mathrm{Diff}(P, \bar{\Lambda}^r) = \mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{*,p+1}(P, \bar{\Lambda}^{q+r})$$

and $E_0^{p,q} \wedge E_0^{1,r} = E_0^{p+1,q+r}$. Since, as we have already shown, $\mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{*,i}(\kappa, \bar{\Lambda}^q)$ and $E_0^{i,q}$, $i = 0, 1$, are isomorphic, the statement we need now follows by induction. ∎

9.4. *The Structure of the Double Complex in* $\mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}\{\kappa\}$

Further, for the definition of the operators $l_{p,q}$ we work on $J^\infty(\pi)$ or within a certain affine chart on N^∞_m. Here $\bar{\Lambda}^i$ is identified with Λ^i_0. This assumption is not necessary for the definition of the operators $\bar{d}_{p,q}$. Thus the double complex structure on $\mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}\{\kappa\}$ described below is defined on $J^\infty(\pi)$ or in an affine chart. The operators

$$l = l_{p,q}\colon \mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{*,p}(\kappa, \bar{\Lambda}^q) \to \mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{*,p+1}(\kappa, \bar{\Lambda}^q)$$

are defined by the "standard" formula

$$\begin{aligned} l(\Delta)(\chi_1,\dots,\chi_{p+1}) &= \sum_i (-1)^{i+1}\, \mathfrak{Z}_{\chi_i}(\Delta(\chi_1,\dots,\hat{\chi}_i,\dots,\chi_{p+1})) \\ &\quad + \sum_{i<j} (-1)^{i+j}\, \Delta([\chi_i,\chi_j],\chi_1,\dots,\hat{\chi}_i,\dots,\hat{\chi}_j,\dots,\chi_{p+1}) \end{aligned} \tag{9.4.1}$$

and, as can be checked directly, have the following properties:

$$\begin{gathered} l_{p+1,q}\circ l_{p,q} = 0, \\ l_{p+k,q+s}(\Delta_1\wedge\Delta_2) = l_{p,q}(\Delta_1)\wedge\Delta_2 + (-1)^{p+q}\Delta_1\wedge l_{k,s}(\Delta_2), \\ \Delta_1 \in \mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{*,p}(\kappa,\bar{\Lambda}^q),\ \Delta_2\in\mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{*,k}(\kappa,\bar{\Lambda}^s). \end{gathered}$$

Also consider the operators

$$\begin{aligned} &\bar{d} = \bar{d}_{p,k}\colon \mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{*,p}(Q,\bar{\Lambda}^k) \to \mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{*,p}(Q,\bar{\Lambda}^{k+1}), \\ &\bar{d}_{p,k}(\Delta)(q_1,\dots,q_p) = (-1)^p\,\bar{d}(\Delta(q_1,\dots,q_p)), \qquad q_i\in Q. \end{aligned} \tag{9.4.2}$$

In view of the fact that the Lie derivative along $\chi\in\kappa$ commutes with the operators $\bar{d}\colon\bar{\Lambda}^q\to\bar{\Lambda}^{q+1}$ it follows immediately from definitions that for $Q=\kappa$

$$\bar{d}_{p+1,q}\circ l_{p,q} + l_{p,q+1}\circ\bar{d}_{p,q} = 0.$$

In other words the systems of operators $\{l_{p,q}\}$ and $\{\bar{d}_{p,q}\}$ transform $\mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}\{\kappa\}$ into a double complex.

Note the following obvious property of the operators $\bar{d}_{p,q}$:

$$\begin{gathered} \bar{d}_{p+k,q+l}(\Delta_1\wedge\Delta_2) = \bar{d}_{p,q}(\Delta_1)\wedge\Delta_2 + (-1)^{p+q}\Delta_1\wedge\bar{d}_{k,l}(\Delta_2), \\ \Delta_1\in\mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{*,p}(\kappa,\bar{\Lambda}^q),\ \Delta_2\in\mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{*,k}(\kappa,\bar{\Lambda}^l) \end{gathered} \tag{9.4.3}$$

Proposition. *Under the isomorphism of Proposition* 9.3 *the operator* $d_0^{p,q}$ *is identified with* $\bar{d}_{p,q}$. *If we also identify, according to* 9.2, $\Lambda^{p,q}$ *with* $\mathscr{C}\,\mathrm{Diff}_{*,p}(\kappa,\bar{\Lambda}^q)$ *then the operators* $d'_{p,q}$ *and* $d''_{p,q}$ *are identified with* $\bar{d}_{p,q}$ *and* $l_{p,q}$, *respectively.*

Proof. Since $d_0 = \{d_0^{p,q}\}$ and $\bar{d} = \{\bar{d}_{p,q}\}$ are derivations in the algebras E_0 and $\mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}\{\kappa\}$, respectively, it suffices to check the first statement for $p = 0$, where it is obvious, and $p = 1$. In the last case, since d_0 and $\bar{d}$ are derivations of the A-modules $\sum_q E_0^{1,q}$ and $\sum_q \mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{*,1}(\kappa, \bar{\Lambda}^q)$, respectively, it suffices to check the statement within an affine chart on elements of the form $U_1(\omega) \in E_0^{1,q}$, $\omega \in \Lambda_0^q$. But for $\bar{\Lambda}_0^*$ we have $U_1 = d - \hat{d}$ (see [13]), while $d'' U_1(\omega) = 0$ according to (9.2.1). Therefore

$$d' U_1(\omega) = dU_1(\omega) = d(d - \hat{d})(\omega) = -(d - \hat{d})(\hat{d}\omega) = -U_1(\hat{d}\omega).$$

But since $\nabla_{U_1(\omega)} = l_\omega$, $\nabla_{U_1(\hat{d}\omega)} = l_{\hat{d}\omega} = \hat{d} \circ l_\omega$ we have $\nabla_{d'U_1(\omega)} = -\hat{d} \circ \nabla_{U_1(\omega)}$. Similarly it suffices to show that the operators $d''_{p,q}$ and $l_{p,q}$ coincide on elements of the form $\rho = U_1(\varphi) \wedge \omega$, $\varphi \in A$, $\omega \in \Lambda_0^q$. But $\nabla_\rho(\mathfrak{Z}_f) = \mathfrak{Z}_f(\varphi)\omega$, $f \in \mathscr{F}(\pi, \pi)$. Therefore since $[\mathfrak{Z}_f, \mathfrak{Z}_g] = \mathfrak{Z}_{\{f,g\}}$,

$$\begin{aligned} l(\nabla_\rho)(f, g) &= \mathfrak{Z}_f(\mathfrak{Z}_g(\varphi)\omega) - \mathfrak{Z}_g(\mathfrak{Z}_f(\varphi)\omega) - \mathfrak{Z}_{\{f,g\}}(\varphi) \cdot \omega \\ &= [\mathfrak{Z}_f, \mathfrak{Z}_g](\varphi) \cdot \omega + \mathfrak{Z}_g(f) \cdot \mathfrak{Z}_f(\omega) \\ &\quad - \mathfrak{Z}_f(\varphi) \cdot \mathfrak{Z}_g(\omega) - \mathfrak{Z}_{\{f,g\}}(\varphi) \cdot \omega \\ &= \mathfrak{Z}_g(\varphi) \cdot \mathfrak{Z}_f(\omega) - \mathfrak{Z}_f(\varphi)\, \mathfrak{Z}_g(\omega). \end{aligned}$$

On the other hand according to (9.2.1), we have $d''\rho = -U_1(\varphi) \wedge U_1(\omega)$ and in an obvious way $\nabla_{d''\rho}(\mathfrak{Z}_f, \mathfrak{Z}_g) = \mathfrak{Z}_g(\varphi)\, \mathfrak{Z}_f(\omega) - \mathfrak{Z}_f(\varphi)\, \mathfrak{Z}_g(\omega)$. ∎

Remark. The complex $\{\mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}\{\kappa\}, l\}$ on $J^\infty(\pi)$ may be identified with the term E_0 of the spectral sequence of de Rham cohomology of the bundle $\pi_\infty\colon J^\infty(\pi) \to M$. This follows from Remark 2 in 9.2 and Proposition 9.3.

9.5. *The Terms $E_1^{p,q}$ in the "Absolute" Case*

Using the isomorphism of Propositions 9.3 and 9.4 we shall compute the term E_1 of the $\mathscr{C}$-spectral sequence as the cohomology of the complex $\{\mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}\{\kappa\}, \bar{d}\}$. Since the complex $\{\mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}\{\kappa\}, \bar{d}\}$ is actually the direct sum of the complexes $\{\mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{(p)}(\kappa), \bar{d}\}$, where $\mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{(p)}(\kappa) = \sum_q \mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{*,p}(\kappa, \bar{\Lambda}^q)$ it suffices to restrict ourselves to the complex $\{\mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{(p)}(\kappa), \bar{d}\}$. The cohomology of this complex can be computed from the following general considerations.

Suppose $Q_1, \dots, Q_k$ are certain A-modules,

$$\mathscr{C}\,\mathrm{Diff}\{Q_1, \dots, Q_k\} = \sum_q \mathscr{C}\,\mathrm{Diff}(Q_1, \dots, Q_k; \bar{\Lambda}^q),$$

where $\mathscr{C}\,\mathrm{Diff}(Q; P) = \mathscr{C}\,\mathrm{Diff}(Q, P)$ and

$$\mathscr{C}\,\mathrm{Diff}(Q_1, \dots, Q_k; P) = \mathscr{C}\,\mathrm{Diff}(Q_1, \mathscr{C}\,\mathrm{Diff}(Q_2, \dots, Q_k; P)).$$

Let us transform $\mathscr{C}\operatorname{Diff}\{Q_1,..., Q_k\}$ into a complex by supplying it with the derivation $\bar{d}$,

$$(\bar{d}\nabla)(q_1,..., q_k) = (-1)^k \bar{d}(\nabla(q_1,..., q_k)), \qquad q_i \in Q_i,$$

where $\nabla(q_1,..., q_k) = (\cdots (\nabla(q_1)(q_2)) \cdots (q_k))$. Starting from the operator $\mu = \mu_1 : \mathscr{C}\operatorname{Diff}(Q_1, \bar{\Lambda}^n) \to \hat{Q}_1$ (see 2.2), we can inductively define the operators

$$\mu = \mu_k : \mathscr{C}\operatorname{Diff}(Q_1,..., Q_k; \bar{\Lambda}^n) \to \mathscr{C}\operatorname{Diff}(Q_1,..., Q_{k-1}; \hat{Q}_k)$$

by putting

$$\mu_k(\nabla) = \mu_{k-1} \circ \nabla.$$

PROPOSITION. *Suppose $A = \mathscr{F}(\mathscr{Y})$ and the A-modules Q_i are projective, $2 \leqslant i \leqslant k$. Then the cohomology of the complex $\{\mathscr{C}\operatorname{Diff}\{Q_1,..., Q_k\}, \bar{d}\}$ are trivial in all dimensions which differ from n and in dimension n are isomorphic to $\mathscr{C}\operatorname{Diff}(Q_1,..., Q_{k-1}; \hat{Q}_k)$.*

Proof. For $k = 1$ the statement is exactly equivalent to the exactness of the Spencer complex (see 7.5). Extend the complex $\mathscr{C}\operatorname{Diff}\{Q_1,..., Q_k\}$ by means of the operator μ_k. Then the statement which is to be proved is equivalent to the acyclicity of the extended complex which can be established by induction over k precisely in the same way as the acyclicity of the complex $\mathscr{S}PQ$ in 3.1. ∎

The permutation group S_k acts in the complex $\{\mathscr{C}\operatorname{Diff}_{(k)}\{Q\}, \bar{d}\}$, where $\mathscr{C}\operatorname{Diff}_{(k)}\{Q\} = \mathscr{C}\operatorname{Diff}\{Q,..., Q\}$ (k times). Indeed, every $\Delta \in \mathscr{C}\operatorname{Diff}_{(k)}\{Q\}$ may be understood as a polydifferential operator (see 3.2)

$$\begin{aligned}(q_1,..., q_k) &\mapsto \Delta(q_1,..., q_k)\\ &= (\cdots (\Delta(q_1)(q_2) \cdots (q_k)) \in \bar{\Lambda}^*, \qquad q_i \in Q.\end{aligned}$$

Then for $\tau \in S_k$ we have

$$\tau(\Delta)(q_1,..., q_k) = \Delta(q_{\tau(1)},..., q_{\tau(k)}).$$

Since the alternation operation determines the complex $\{\mathscr{C}\operatorname{Diff}^{\mathrm{alt}}_{(k)}\{\kappa\}, \bar{d}\}$ as a direct summand in $\{\mathscr{C}\operatorname{Diff}_{(k)}\{\kappa\}, \bar{d}\}$ we see that its cohomology is trivial in all dimensions which differ from n while in dimension n they are isomorphic to the anti-symmetric part of the A-module $\mathscr{C}\operatorname{Diff}_{*,k-1}(\kappa; \hat{\kappa})$ (see 3.2), denoted by $L_k(\kappa)$, with respect to the induced action of the group S_k which will be described further.

9.6. *The Action of* S_k *in* $\mathscr{C}\,\mathrm{Diff}_{*,k-1}(P,\hat{P})$

The inclusion

$$i_1\colon \mathscr{C}\,\mathrm{Diff}_{*,0}(P;\hat{P}) = \hat{P} = \mathrm{Hom}_A(P,\bar{\Lambda}^n) \hookrightarrow \mathscr{C}\,\mathrm{Diff}(P,\bar{\Lambda}^n)$$

induces the inclusions

$$\begin{aligned} i_2\colon \mathscr{C}\,\mathrm{Diff}_{*,1}(P;\hat{P}) &= \mathscr{C}\,\mathrm{Diff}(P,\mathrm{Hom}(P,\bar{\Lambda}^n)) \\ &\to \mathscr{C}\,\mathrm{Diff}(P,\mathscr{C}\,\mathrm{Diff}(P,\bar{\Lambda}^n)) = \mathscr{C}\,\mathrm{Diff}_{*,2}(P;\bar{\Lambda}^n) \\ &\;\vdots \\ i_k\colon \mathscr{C}\,\mathrm{Diff}_{*,k-1}(P;\hat{P}) &\to \mathscr{C}\,\mathrm{Diff}_{*,k}(P;\bar{\Lambda}^n). \end{aligned}$$

Here we have $\mu_k \circ i_k = \mathrm{id}$ and $\mathrm{im}\, i_k$ is invariant with respect to the action of the sugroup $S_{k-1} \subset S_k$ which leaves the kth index in its place. For this reason

$$L_k(P) \subset \mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{*,k-1}(P;\hat{P}).$$

Now let us describe the action of the transposition $\tau \in S_k$ which interchanges the ith and the kth indices, $i < k$.

Let us fix the elements $p_1,\dots,p_{i-1},\ p_{i+1},\dots,p_{k-1} \in P$ and consider the operator $\square \in \mathrm{im}\, i_2$, $\square(p_i)(p_k) = \Delta(p_1,\dots,p_k)$, where $\Delta \in \mathrm{im}\, i_k$. Suppose $\tilde{\square}(p_i)(p_k) = \square(p_k)(p_i)$. Then it follows from Green's $\mathscr{C}$-formula that

$$\tilde{\square}(p_i)(p_k) = \langle \square(p_k), p_i \rangle = \langle p_k, \tilde{\square}^*(p_i) \rangle + \bar{d}v(p_1,\dots,p_k),$$

where v is some polydifferential operator. The polydifferential operator $\Delta'\colon (p_1,\dots,p_k) \mapsto \langle p_k, \tilde{\square}^*(p_i) \rangle$ belongs to the image of i_k. Since $\tau(\Delta)(p_1,\dots,p_k) = \tilde{\square}(p_i)(p_k)$ and $\ker \mu_k = \mathrm{im}\, \bar{d}$ we have $\Delta' = i_k(\tau(\nabla))$ if $\Delta = i_k(\nabla)$. This proves the following:

PROPOSITION. *Suppose* $\tau \in S_k$ *is a transposition which interchanges the* ith *and the* kth *elements,* $i < k$, *and* $\nabla \in \mathscr{C}\,\mathrm{Diff}_{*,k-1}(P;\hat{P})$. *Then*

$$\tau(\nabla)(p_1,\dots,p_{k-1}) = \square_i(p_1,\dots,p_{k-1})^*\,(p_i),$$

where $\square_i(p_1,\dots,p_{k-1}) \in \mathscr{C}\,\mathrm{Diff}(P,\hat{P})$,,

$$\square_i(p_1,\dots,p_{k-1})(p) = \nabla(p_1,\dots,p_{i-1},p,p_{i+1},\dots,p_{k-1}).$$

COROLLARY. $L_k(P)$ *consists of all those* $\nabla \in \mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{*,k-1}(P;\hat{P})$ *such that* $\square_i(p_1,\dots,p_{k-1})^* = -\square_i(p_1,\dots,p_{k-1})$ *for all* $1 \leqslant i \leqslant k-1$, $p_1,\dots,p_{k-1} \in P$.

9.7. *The Description of $\{E_r^{p,q}, d_r\}$ in the "Absolute" Case*

Let us bring together all the results obtained above.

THEOREM. *If $\mathscr{Y} \subset J^\infty(\pi)$ or N_m^∞ is open, then*

(1) $E_1^{0,q} = \bar{H}^q(\mathscr{Y})$; $E_1^{p,q} = 0$ *if* $p > 0$, $q \neq n$; $E_1^{p,n} = L^p(\kappa)$, $p > 0$ *(we assume that $L_1(\kappa) = \hat{\kappa}$);*

(2) $E_1^{0,q} = E_\infty^{0,q}$, $q < n$; $E_2^{p,n} = E_\infty^{p,n}$, $p \geqslant 0$; $E_r^{p,q} = 0$, $2 \leqslant r \leqslant \infty$ *if* $p > 0$, $q \neq n$, *or* $p = 0$, $q > n$.

Proof. The proof follows immediately from the description of the terms E_0 and E_1 given above. ∎

Let us put $L_0(\kappa) = \bar{H}^n(\mathscr{Y})$ and introduce the operators $\mathscr{E}_i$: $L_i(\kappa) \to L_{i+1}(\kappa)$ which reduce to $d_1^{i,n}$ under the identification of $E_1^{i,n}$ with $L_i(\kappa)$ described above.

COROLLARY. (1) *If* $q < n$ *then* $H^q(\mathscr{Y}) = \bar{H}^q(\mathscr{Y})$; *if* $q \geqslant n$ *then* $H^q(\mathscr{Y}) = E_2^{q-n,n}$ *or which is the same thing* $H^q(\mathscr{Y})$ *is isomorphic to the* $(q-n)$th *cohomology group of the complex* $\{L_i, \mathscr{E}_i\}$.

(2) *If* $\mathscr{Y} = J^\infty(\pi)$ *then* $H^q(J^\infty(\pi)) = H^q(J^0(\pi))$; *if* $\mathscr{Y} = N_m^\infty$ *then* $H^q(N_m^\infty) = H^q(N_m^1)$.

Proof. (1) Immediately follows from the theorem given above, while (2) follows from the fact that the fibers of the projection $\pi_{k,k-1}$: $J^k(\pi) \to J^{k-1}(\pi)$ are contractible for $k \geqslant 1$ as are the fibres of the projections $\pi_{k,k-1}$: $N_m^k \to N_m^{k-1}$ for $k \geqslant 2$ (see, e.g., [13]) since Λ^* is the direct limit of an increasing chain of complexes $\pi_{\infty,k}^*(\Lambda^*(\mathscr{Y}_k))$, where $\mathscr{Y}_k = J^k(\pi)$ or N_m^k, while $\pi_{\infty,k}^*$ are monomorphisms. ∎

Remark. The manifold N_m^1 is obviously the manifold of all the tangent n-dimensional subspaces to the manifold N. Therefore the fiber of the projection $\pi_{1,0}$: $N_m^1 \to N_m^0 = N$ is the Grassman manifold $G_{m+n,n}$ and the cohomology of the manifold N_m^1 may be computed by using the Leray–Hirsch theorem (see [37]).

9.8. *Relationship with the Calculus of Variations and the Computation of the Operators $\mathscr{E}_i$*

In this subsection we give an explicit description of the differential $d_1^{i,n}$ or which is equivalent of the differential $\mathscr{E}_i$. The relationship between the $\mathscr{C}$-spectral sequence and the calculus of variations consists first of all in the fact that the differential $d_1^{0,n}$: $E_1^{0,n} = H^n(\mathscr{Y}) \to E_1^{1,n} = \hat{\kappa}$ coincides in any affine chart, as will be shown below, with the Euler operator $\mathscr{E}$ (see 8.3). Thus $d_1^{0,n}$ may be viewed as the "global" Euler operator (see 8.3). The

following proposition is a direct consequence of the above and the fact that the $\mathscr{C}$-spectral sequence is natural in DE.

PROPOSITION. *The Euler operator is natural in the category* DE.

Remark. Suppose Δ is a nonlinear differential operator from $\Gamma(\pi_2)$ to $\Gamma(\pi_1)$, where $\pi_i: E_{\pi_i} \to M$, $i = 1, 2$, are smooth bundles. It generates naturally a map $F: J^\infty(\pi_2) \to J^\infty(\pi_1)$, where $F^*: \mathscr{F}(\pi_1) \to \mathscr{F}(\pi_2)$ is a morphism in DE. Similarly a nonlinear differential operator Δ given on n-dimensional submanifolds of the manifold N_2 and assuming its values on n-dimensional submanifolds of the manifold N_1 generates a morphism $F^*: \mathscr{F}_{m_1}(N_1) \to \mathscr{F}_{m_2}(N_2)$ in the category DE, where $\dim N_i = n + m_i$. Then Proposition 9.8 means that

$$\mathscr{E} \circ F_{\bar{H}}^* = F_\kappa^* \circ \mathscr{E},$$

where $F_{\bar{H}}^*: \bar{H}^*(\pi_1) \to \bar{H}^*(\pi_2)$ (resp. $F_{\bar{H}}^*: \bar{H}^*((N_1)_{m_1}^\infty) \to \bar{H}^*((N_2)_{m_2}^\infty)$) and $F_\kappa^*: \hat{\kappa}(\pi_1) \to \hat{\kappa}(\pi_2)$ (resp. $F_\kappa^*: \hat{\kappa}((N_1)_{m_1}^\infty) \to \kappa((N_2)_{m_2}^\infty)$) are the maps induced by F. This shows that this proposition strengthens the well-known fact that the Euler equation is invariant under changes of dependent and independent variables which correspond to the zeroth order operator Δ in the situation described above.

It follows from Propositions 9.3 and 9.4 and the general theory of double complexes (see [38]) that the composition $E_0^{0,n} = \bar{\Lambda}^n \to E_1^{0,n} = \bar{H}^n(\mathscr{Y}) \to^{d_1^{0,n}} E_1^{1,n} = \hat{\kappa}$ coincides under the identifications above with $\mu \circ l_{0,n}$ while $d_1^{p,n} = \mathscr{E}_p$ coincides with $\mu_{p+1} \circ l_{p,n} \circ i_p^{\mathrm{alt}}$, where i_p^{alt} is the composition of i_p with the alternation operation (see 9.5).

Further we work on $J^\infty(\pi)$ or, which is the same thing, within an affine chart on N_m^∞. According to subsection 6.6 we can identify $\kappa = \kappa(\pi)$ with $\mathscr{F}(\pi, \pi)$ and for the polydifferential operators ∇ on κ we shall write $\nabla(f_1, \ldots, f_p)$ instead of $\nabla(\mathfrak{Z}_{f_1}, \ldots, \mathfrak{Z}_{f_p})$.

As was noted above, $l_{0,k}(\omega) = l_\omega$, $\omega \in \bar{\Lambda}^k$. Therefore

$$(\mu \circ l_{0,k})(\omega) = l_\omega^*(1) = \mathscr{E}(\mathscr{L}), \qquad \mathscr{L} = \langle \omega \rangle,$$

so that we have established the fact that $\mathscr{E}$ and $d_1^{0,n}$ coincide. To compute the operators $d_1^{p,n}$ we shall use the following:

LEMMA. *Suppose* $\square \in \mathscr{C}\,\mathrm{Diff}_{*,p}^{\mathrm{alt}}(\kappa, \bar{\Lambda}^n)$, *then*

$$\begin{aligned}[(\mu_{p+1} \circ l_{p,n})(\square)](f_1, \ldots, f_p) &= (-1)^p\, l_{\square(f_1,\ldots,f_p)}^*(1) + \sum_{s \leqslant p} (-1)^{s-1}\, \mathfrak{Z}_{f_s}(\square_s^*(1)) \\ &\quad + \sum_{s \leqslant p} (-1)^{s-1}\, l_{f_s}^*(\square_s^*(1)) + \sum_{i<j \leqslant p} (-1)^{i+j}\, \square_{ij}^*(1), \qquad (9.8.1)\end{aligned}$$

where the operators $\square_s, \square_{ij} \in \mathscr{C}\,\mathrm{Diff}(\kappa, \bar{\Lambda}^n)$ are defined by the relations $\square_s(f) = \square(f_1,\ldots,\hat{f}_s,\ldots,f_p,f)$ and $\square_{ij}(f) = (\{f_i,f_j\}, f_1,\ldots,\hat{f}_i,\ldots,\hat{f}_j,\ldots,f_p,f)$.

Proof. According to (9.4.1) we have

$$\begin{aligned} l_{p,n}(\square)(f_1,\ldots,f_{p+1}) &= \sum_{s\leqslant p} (-1)^{s-1}\,\mathfrak{Z}_{f_s}(\square(f_1,\ldots,\hat{f}_s,\ldots,f_{p+1})) \\ &\quad + (-1)^p\,\mathfrak{Z}_{f_{p+1}}(\square(f_1,\ldots,f_p)) \\ &\quad + \sum_{i<j\leqslant p} (-1)^{i+j}\,\square(\{f_i,f_j\}, f_1,\ldots,\hat{f}_i,\ldots,\hat{f}_j,\ldots,f_{p+1}) \\ &\quad + \sum_{s\leqslant p} (-1)^{s+p+1}\square(\{f_s,f_{p+1}\}, f_1,\ldots,\hat{f}_s,\ldots,f_p) \\ &= \sum_{s\leqslant p} (-1)^{s-1}\,\{[\mathfrak{Z}_{f_s},\square_s](f_{p+1}) + \square_s(\mathfrak{Z}_{f_s}(f_{p+1}))\} \\ &\quad + (-1)^p\,l_{\square(f_1,\ldots,f_p)}(f_{p+1}) + \sum_{i<j\leqslant p} (-1)^{i+j}\,\square_{ij}(f_{p+1}) \\ &\quad + \sum_{s\leqslant p} (-1)^s\,\{\square_s(\mathfrak{Z}_{f_s}(f_{p+1})) - \square_s(\mathfrak{Z}_{f_{p+1}}(f_s))\} \\ &= \sum_{s\leqslant p} (-1)^{s-1}\,[\mathfrak{Z}_{f_s},\square_s](f_{p+1}) + (-1)^p l_{\square(f_1,\ldots,f_p)}(f_{p+1}) \\ &\quad + \sum_{i<j\leqslant p} (-1)^{i+j}\,\square_{ij}(f_{p+1}) + \sum_{s\leqslant p} (-1)^{s-1}\,\square_s(l_{f_s}(f_{p+1})). \end{aligned}$$

If we introduce the operator

$$\begin{aligned} \nabla = \nabla_{f_1,\ldots,f_p} &= \sum_{s\leqslant p} (-1)^{s-1}\,[\mathfrak{Z}_{f_s},\square_s] + (-1)^p\,l_{\square(f_1,\ldots,f_p)} \\ &\quad + \sum_{i<j\leqslant p} (-1)^{i+j}\,\square_{ij} + \sum_{s\leqslant p} (-1)^{s-1}\,\square_s \circ l_{f_s} \end{aligned}$$

the result of the computation carried out above may be represented in the form $l_{p,n}(\square)(f_1,\ldots,f_{p+1}) = \nabla(f_{p+1})$. But

$$\nabla(f_{p+1}) = (\nabla^*(1), f_{p+1}) + \bar{d}\mathfrak{K}_\lambda(\nabla \circ f_{p+1}).$$

But since the polydifferential operator $(f_1,\ldots,f_{p+1}) \mapsto (\nabla^*(1), f_{p+1})$ is a homomorphism with respect to the last argument, it belongs to the image of i_{p+1} and therefore

$$[(\mu_{p+1} \circ l_{p,n})(\square)](f_1,\ldots,f_p) = \nabla^*_{f_1,\ldots,f_p}(1). \quad \blacksquare$$

Now suppose $\Delta \in L_p(\kappa)$, $p \geqslant 1$. If $\square = i_p^{\mathrm{alt}}(\Delta)$ then

$$\square(f_1,\ldots,f_p) = \frac{1}{p} \sum_{i\leqslant p} (-1)^{p-i}\,\Delta(f_1,\ldots,\hat{f}_i,\ldots,f_p)(f_i). \tag{9.8.2}$$

To compute $\mathscr{E}_p(\Delta)$ let us use the lemma proved above and express the aggregates $\Box_s^*(1)$, $\Box_{ij}^*(1)$, $l^*_{\Box(f_1,\ldots,f_p)}(1)$ in terms of Δ. Introduce the operators $\Delta_{is} \in \mathscr{C}\,\mathrm{Diff}(\kappa, \hat{\kappa})$ by putting $\Delta_{is}(f_s) = \nabla(f_1,\ldots,\hat{f}_i,\ldots,f_p)$. Then, as can be checked immediately from definitions,

$$\Box_s = \frac{1}{p} \sum_{i \leqslant p} (-1)^{i+s} f_i^* \circ \Delta_{is}.$$

Since $\Delta \in L_p(\kappa)$ we have $\Delta_{is}^* = -\Delta_{is}$ according to Corollary 9.6 and therefore

$$\begin{aligned}\Box_s^*(1) &= \frac{1}{p} \sum_{i \leqslant p} (-1)^{i+s} (\Delta_{is}^* \circ f_i)(1) = \frac{1}{p} \sum_{i \leqslant p} (-1)^{i+s} \Delta_{is}^*(f_i) \\ &= \frac{1}{p} \sum (-1)^{i+s-1} \Delta_{is}(f_i) = \Delta(f_1,\ldots,\hat{f}_s,\ldots,f_p).\end{aligned}$$

Similarly, $\Box_{ij}^*(1) = \Delta(\{f_i, f_j\}, f_1,\ldots,\hat{f}_i,\ldots,\hat{f}_j,\ldots,f_p)$. Further since the operator $\varphi \mapsto l_\varphi$ is a derivation (see 6.6) we have $l_{(\varphi,\psi)} = \varphi^* \circ l_\psi + \psi^* \circ l_\varphi$, where $\varphi \in P$, $\psi \in \hat{P}$ and $l^*_{(\varphi,\psi)}(1) = l_\psi^*(\varphi) + l_\varphi^*(\psi)$. Applying this to (9.8.2) we find that

$$\begin{aligned}l^*_{\Box(f_1,\ldots,f_p)}(1) = \frac{1}{p} \sum_{i \leqslant p} (-1)^{p-i} \, [(l^*_{\Delta(f_1,\ldots,\hat{f}_i,\ldots,f_p)}(f_i) \\ + l_{f_i}^*(\Delta(f_1,\ldots,\hat{f}_i,\ldots,f_p))].\end{aligned}$$

Substituting the expression that we have found into (9.8.1) we obtain for

$$\begin{aligned}&\mathscr{E}_p(\Delta)(f_1,\ldots,f_p) \\ &\quad = \sum_{s \leqslant p} (-1)^{s-1}\, \mathfrak{Z}_{f_s}(\Delta(f_1,\ldots,\hat{f}_s,\ldots,f_p)) \\ &\qquad + \sum_{s<k \leqslant p} (-1)^{s+k} \Delta(\{f_s, f_k\}, f_1,\ldots,\hat{f}_s,\ldots,\hat{f}_k,\ldots,f_p) \qquad (9.8.3) \\ &\qquad + \frac{1}{p} \sum_{s \leqslant p} (-1)^{s-1} \, [(p-1)\, l_{f_s}^*(\Delta(f_1,\ldots,\hat{f}_s,\ldots,f_p)) - l^*_{\Delta(f_1,\ldots,\hat{f}_s,\ldots,f_p)}(f_s)].\end{aligned}$$

For $p = 1$ this formula yields

$$\mathscr{E}_1(\varphi)(f) = \mathfrak{Z}_f(\varphi) - l_\varphi^*(f) = (l_\varphi - l_\varphi^*)(f),$$

where $\varphi \in \hat{\kappa}$, i.e.,

$$\mathscr{E}_1(\varphi) = l_\varphi - l_\varphi^*. \qquad (9.8.4)$$

9.9. *The Resolvent for the Euler Operator*

If $\mathscr{Y}$ is open in N_m^∞ (or in $J^\infty(\pi)$), the description of the $\mathscr{C}$-spectral sequence for $\mathscr{Y}$ given in 9.7 may be restated briefly as follows. Consider the following complex which will further be called the variation complex,

$$0 \to A = \bar{\Lambda}^\sigma \xrightarrow{\bar{d}} \cdots \xrightarrow{\bar{d}} \bar{\Lambda}^n \xrightarrow{\mathscr{E}'} L_1 \xrightarrow{\mathscr{E}_1} L_2 \xrightarrow{\mathscr{E}_2} \cdots, \tag{9.9.1}$$

where $\mathscr{E}'$ is the composition of the natural projection $\bar{\Lambda}^n \to \bar{H}^n(\mathscr{Y})$ and $\mathscr{E}_0 = d_1^{0,n}$. In view of Theorem 9.7 the k-dimensional cohomology of this complex, if we count from the left, coincides with $H^k(\mathscr{Y})$. Therefore its fragment, to the right of $\bar{\Lambda}^n$ as well as the complex $\{L_i, \mathscr{E}_i\}$, is a resolvent for the Euler operator $\mathscr{E}$ if $H^k(\mathscr{Y}) = 0$ for $k > n$. Locally this is always to. Note the following important special case.

COROLLARY. *If π is a vector bundle then $\{L_i, \mathscr{E}_i\}$ is the resolvent for the Euler operator on $J^\infty(\pi)$.*

Let us say that the Lagrangian $\mathscr{L} \in \bar{H}^n(\mathscr{Y})$ is trivial if $\mathscr{E}(\mathscr{L}) = 0$. The vector space of all trivial Lagrangians obviously coincides with $E_2^{0,n}$ and therefore, by Theorem 9.7, we have:

PROPOSITION. *The vector space of trivial Lagrangians is isomorphic to $H^n(\mathscr{Y})$. In particular, if $H^n(\mathscr{Y}) = 0$, then a Lagrangian is trivial if and only if its density is trivial.*

The simplest application of the variational complex consists of the fact that it gives a method for checking if the given determined system of equations is an Euler–Lagrange system for some Lagrangian. Every such system, at least in a homotopically trivial situation, may be represented in the form $\varphi = 0$, where $\varphi \in \hat{\kappa}$. Hence up to certain easily verified topological conditions such a system is an Euler–Lagrange system whenever $\mathscr{E}_1(\varphi) = 0$. Note the following particular case:

THEOREM. *Suppose $\pi \colon E_\pi \to M$ is a bundle such that $H^{n+1}(E_\pi) = 0$. Then the system of equations $\varphi = 0$, $\varphi \in \hat{\kappa}(\pi)$, is an Euler–Lagrange system whenever $l_\varphi^* = l_\varphi$.*

Proof. This follows from what have just said above and (9.8.4). ∎

It is natural to call a nonlinear system self-adjoint if it is of the form $\varphi = 0$ and $l_\varphi^* = l_\varphi$. If we accept such a terminology we immediately see the relationship between the above theorem and subsection 3.3.

If the system $\varphi = 0$, $\varphi \in \hat{\kappa}$, is an Euler–Lagrange system then $F(\varphi) = \mathscr{E}(\mathscr{L})$, where $F \in \operatorname{Aut} \hat{\kappa}$. Therefore locally under the assumptions of the previous theorem the relation $l_{F(\varphi)} = l_{F(\varphi)}^*$ for some $F \in \operatorname{Aut} \hat{\kappa}$ is a

necessary and sufficient condition for the system $\varphi = 0$ to be an Euler–Lagrange system. This gives a practically convenient method for deciding this question for concrete systems.

Remark. An interesting resolvent of the operator $\mathscr{E}$ for the case of polynomial Lagrangians on the real line was constructed by Olver and Shakiban [39]. It differs from ours. In particular, the "resolving" operator for $\mathscr{E}$ which they construct has in our notations the form $\varphi \mapsto l_\varphi(u) - l_\varphi^*(u)$, where $u = p_\varnothing$ is the coordinate function on $J^1(\pi)$ while π is the trivial one-dimensional bundle over $\mathbb{R}^1$. This means that in the class of polynomial φ the condition $l_\varphi^* = l_\varphi$ is equivalent to the condition $l_\varphi^*(u) = l_\varphi(u)$ although in general this is not so (e.g., for $\varphi = p_1 u^{-2}$).

9.10. *The Infinitesimal Stokes Formula in* E_1

If $\Delta \in L_i$ and $\mathscr{E}_i(\Delta) = 0$, then when there are no topological obstructions, i.e., when $H^i(\mathscr{Y}) = 0$ we have $\Delta = \mathscr{E}_{i-1}(\Delta')$. Our goal is now to give explicit formulas ("inversion formulas") for finding Δ' when Δ is given. This formula will be a corollary to the "infinitesimal Stokes formula" which as will be shown below may be reduced to the term E_1 of the $\mathscr{C}$-spectral sequence.

Suppose the algebra A is an object of DE. Then every Lie derivative operator along the field $X \in D_{\mathscr{C}}(A)$ preserves the ideals $\mathscr{C}^k(A)$ and therefore generates an endomorphism of the corresponding $\mathscr{C}$-spectral sequence. Let us also refer to it as the Lie derivative (on $E_r^{p,q}$). Introduce the notation $e \mapsto X\{e\}$, $e \in E_r^{p,q}$.

Suppose $X \in \mathscr{C}D(A)$. Then

$$X \lrcorner \mathscr{C}\Lambda^* = X \lrcorner (\mathscr{C}\Lambda^1 \wedge \Lambda^*) \subset \mathscr{C}\Lambda^1 \wedge \Lambda^* = \mathscr{C}\Lambda^*$$

and therefore $X \lrcorner \mathscr{C}^k\Lambda^* \subset \mathscr{C}^k\Lambda^*$. Similarly for $X \in D(A)$ we have $X \lrcorner \mathscr{C}^k\Lambda^* \subset \mathscr{C}^{k-1}\Lambda^*$. If $\omega \in \mathscr{C}^p\Lambda^*$ and $d\omega \in \mathscr{C}^{p+k}\Lambda^*$ it follows from the formula $X(\omega) = X \lrcorner d\omega + d(X \lrcorner \omega)$ that $X(\omega) \equiv d\rho \mod \mathscr{C}^{p+k}\Lambda^*$, where $\rho \in \mathscr{C}^p\Lambda^*$, if $X \in \mathscr{C}D(A)$. In other words, $X\{e\} = 0$ when $k > 0$, where the element $e \in E_k$ is represented by the form ω. Thus the Lie derivative along $X \in \mathscr{C}D(A)$ in the $\mathscr{C}$-spectral sequence is always trivial whenever $k > 0$. Thus we have defined the action of the elements of the quotient space $D_{\mathscr{C}}(A)/\mathscr{C}D(A)$ on $E_r^{p,q}$, $r > 0$. This action will also be called a Lie derivation. Thus if $A = \mathscr{F}(\mathscr{Y}_\infty)$, $\mathscr{F}_m(N)$ or $\mathscr{F}(\pi)$ a Lie derivation $\chi\{e\}$ is defined where $e \in E_r^{p,q}$ and $\chi \in \operatorname{Sym} \mathscr{Y}$, $\chi \in \kappa(N_m^\infty)$ or $\chi \in \kappa(\pi)$, respectively.

For $X \in D_{\mathscr{C}}(A)$ we also have a substitution operation $X \lrcorner e \in E_1^{p-1,q}$, where $e \in E_1^{p,q}$. Indeed suppose $\omega \in \mathscr{C}^p\Lambda^{p+q}$ is a $d_0^{p,q}$-cocycle which represents the class e, i.e.,

$$e = \omega + d(\mathscr{C}^p\Lambda^{p+q-1}) + \mathscr{C}^{p+1}\Lambda^{p+q}.$$

Then $X \lrcorner\, \omega \in \mathscr{C}^{p-1}\Lambda^{p+q-1}$ and $d(X \lrcorner\, \omega) = X(\omega) - X \lrcorner\, d\omega \in \mathscr{C}^p\Lambda^{p+q}$, i.e., $X \lrcorner\, \omega$ is a $d_0^{p-1,q}$-cocycle. If $\rho \in \mathscr{C}^p\Lambda^{p+q-1}$, $\lambda \in \mathscr{C}^{p+1}\Lambda^{p+q}$ then

$$X \lrcorner\, (d\rho + \lambda) = X(\rho) - d(X \lrcorner\, \rho) + X \lrcorner\, \lambda \in \mathscr{C}^p\Lambda^{p+q-1} + d\mathscr{C}^{p-1}\Lambda^{p+q-2}.$$

Thus the d_0-cohomology class of the d_0-cocycle $X \lrcorner\, \omega$ is well defined by the class e. Denote it by $X \lrcorner\, e$.

If in the previous arguments we assume that $X \in \mathscr{C}D(A) \subset D_{\mathscr{C}}(A)$, in view of the fact that $X \lrcorner\, \mathscr{C}^p\Lambda^* \subset \mathscr{C}^p\Lambda^*$ they will imply that $X \lrcorner\, e = 0$. This gives us the possibility of defining the substitution operation $\chi \lrcorner\, e \in E_1^{p-1,q}$, where $e \in E_1^{p,q}$ while $\chi \in D_{\mathscr{C}}(A)/\mathscr{C}D(A)$. Now taking the d_0-cohomology classes of each of the summands in the Stokes infinitesimal formula $X(\omega) = X \lrcorner\, d\omega + d(X \lrcorner\, \omega)$ we obtain the following infinitesimal Stokes formula in E_1:

$$\chi(e) = d_1^{p-1,q}(\chi \lrcorner\, e) + \chi \lrcorner\, d_1^{p,q}(e), \qquad e \in E_1^{p,q}, \quad \chi \in D_{\mathscr{C}}(A)/\mathscr{C}D(A). \tag{9.10.1}$$

When $\mathscr{Y}$ is open in $J^\infty(\pi)$ or N_m^∞ the Lie derivation operation and the substitution operation of the elements $\chi \in \kappa(\mathscr{Y})$ may be carried over from $E_1^{p,n}$ into $L_p(\kappa)$ by using the identifications described in 9.6. Let us keep the same notations for the new operations thus carried over. Then (9.10.1) implies

$$\chi\{\Delta\} = \mathscr{E}_{p-1}(\chi \lrcorner\, \Delta) + \chi \lrcorner\, \mathscr{E}_p(\Delta), \qquad \Delta \in L_p(\kappa). \tag{9.10.2}$$

Now let us compute the operations $\chi \lrcorner$ and $\chi\{\cdot\}$ explicitly. Suppose $\Delta \in L_p(\kappa)$ and the operator $\square \in \mathscr{C}\ \mathrm{Diff}^{\mathrm{alt}}_{*,p}(\kappa, \bar{\Lambda}^n)$ is defined by formula (9.8.2). Then it follows from (9.3.1) and the method used for identifying L_p with $E_1^{p,n}$ that

$$\chi \lrcorner\, \Delta = \mu_{p-1}(\chi \lrcorner\, \square),$$

where $\square \in \mathscr{C}\ \mathrm{Diff}^{\mathrm{alt}}_{*,p-1}(\kappa, \bar{\Lambda}^n)$ and

$$(\chi \lrcorner\, \square)(\chi_1, \dots, \chi_{p-1}) = \square(\chi, \chi_1, \dots, \chi_{p-1}).$$

But

$$\begin{aligned} p \cdot (\chi \lrcorner\, \square)(\chi_1, \dots, \chi_{p-1}) &= (-1)^{p-1}\, \Delta(\chi_1, \dots, \chi_{p-1})(\chi) \\ &\quad + \sum_{i=2}^{p-2} (-1)^{p-i-1}\, \Delta(\chi, \chi_1, \dots, \hat{\chi}_i, \dots, \chi_{p-1})(\chi_i) \\ &\quad + \Delta(\chi, \chi_1, \dots, \chi_{p-2})(\chi_{p-1}) = (-1)^{p-1}\, (\chi^* \circ \Delta_{\chi_1, \dots, \chi_{p-2}})(\chi_{p-1}) \end{aligned}$$

$$+\sum_{i=2}^{p-2}(-1)^{p-i-1}(\chi_i^*\circ\Delta_{\chi,\ldots,\hat{\chi}_i,\ldots,\chi_{p-2}})(\chi_{p-1})$$
$$+\Delta(\chi,\chi_1,\ldots,\chi_{p-2})(\chi_{p-1}),$$

where

$$\Delta_{\chi_1,\ldots,\chi_{p-2}}(\chi_{p-1})=\Delta(\chi_1,\ldots,\chi_{p-1}),$$
$$\Delta_{\chi,\ldots,\hat{\chi}_i,\ldots,\chi_{p-2}}(\chi_{p-1})=\Delta(\chi,\chi_1,\ldots,\hat{\chi}_i,\ldots,\chi_{p-1}).$$

Now using Green's $\mathscr{C}$-formula and Corollary 9.6 we obtain

$$\begin{aligned}(-1)^{p-1}&(\chi^*\circ\Delta_{\chi_1,\ldots,\chi_{p-2}})(\chi_{p-1})\\ &=(-1)^{p-1}\langle\Delta^*_{\chi_1,\ldots,\chi_{p-2}}(\chi),\chi_{p-1}\rangle+\bar{d}v(\chi,\chi_1,\ldots,\chi_{p-1})\\ &=(-1)^p(\Delta_{\chi_1,\ldots,\chi_{p-2}}(\chi))(\chi_{p-1})+\bar{d}v\\ &=\Delta(\chi,\chi_1,\ldots,\chi_{p-2})(\chi_{p-1})+\bar{d}v,\end{aligned}$$

where $v\in\mathscr{C}\,\mathrm{Diff}_{*,p-1}(\kappa,\bar{\Lambda}^{n-1})$, and similarly

$$(-1)^{p-i-1}(\chi_i^*\circ\Delta_{\chi,\ldots,\hat{\chi}_i,\ldots,\chi_{p-2}})(\chi_{p-1})=\Delta(\chi,\chi_1,\ldots,\chi_{p-2})(\chi_{p-1})+\bar{d}v_i.$$

Therefore

$$(\chi\,\lrcorner\,\Delta)(\chi_1,\ldots,\chi_{p-2})=\Delta(\chi,\chi_1,\ldots,\chi_{p-2}).\tag{9.10.3}$$

An explicit formula for $\chi\{\Delta\}$ can now be obtained directly from (9.10.2), (9.10.3), and (9.8.3),

$$\begin{aligned}\chi\{\Delta\}&(\chi_1,\ldots,\chi_{p-1})\\ &=\mathrm{Э}_\chi(\Delta(\chi_1,\ldots,\chi_{p-1}))+\sum_i\Delta(\chi_1,\ldots,[\chi_i,\chi],\ldots,\chi_{p-1})\\ &\quad+\frac{1}{p(p-1)}\sum_i(-1)^i\,[l^*_{\Delta(\chi,\chi_1,\ldots,\hat{\chi}_i,\ldots,\chi_{p-1})}(\chi_i)\\ &\quad+l^*_{\chi_i}(\Delta(\chi,\chi_1,\ldots,\hat{\chi}_i,\ldots,\chi_{p-1}))]\\ &\quad+\frac{p-1}{p}\,l^*_\chi(\Delta(\chi_1,\ldots,\chi_{p-1}))-\frac{1}{p}\,l^*_{\Delta(\chi_1,\ldots,\chi_{p-1})}(\chi).\end{aligned}\tag{9.10.4}$$

Since the formula above presuppose that $p>1$ the case $p=1$ must be considered separately. Suppose $\varphi\in\hat{\kappa}=L_1(\kappa)$. Then $\chi\,\lrcorner\,\varphi=(\varphi,\chi)$ and as we

have already seen (see 9.8) $l_{(\varphi,\chi)} = \varphi^* \circ l_\chi + \chi^* \circ l_\varphi$. Therefore (9.10.2), (9.8.4), and the fact that $\mathscr{E}_0(\langle\omega\rangle) = l_\omega^*(1)$, $\omega \in \bar{\Lambda}^n$, imply

$$\chi\{\varphi\} = l_\varphi(\chi) + l_\varphi^*(\chi) = 3_\chi(\varphi) + l_\varphi^*(\chi). \tag{9.10.5}$$

It should be noted that the natural action of the elements $\chi \in \kappa$ on $L_i(\kappa)$ as on polydifferential operators, which are easy to define, by using the considerations in 1.5 differ from the ones considered above.

9.11. *The Inversion Formulas*

Suppose $G_t\colon \mathscr{Y} \to \mathscr{Y}$ is a smooth family of automorphisms of a certain object of DE, $t \in [0, 1]$, and $X_t \in D_{\mathscr{C}}(\mathscr{Y})$ is the corresponding family of $\mathscr{C}$-fields, i.e.,

$$\frac{d}{dt}(G_t^*) = G_t^* \circ X_t.$$

If $d_1^{p,q}(e) = 0$ and $\chi_t = X_t \bmod \mathscr{C}D(\mathscr{Y})$ then by (9.10.1) we get

$$\frac{d}{dt}(G_t^*(e)) = G_t^*(\chi_t\{e\}) = G_t^*(d_1(\chi_t \lrcorner e)) = d_1 G_t^*(\chi_t \lrcorner e).$$

Integrating this equality with respect to t we obtain if

$$G_0 = \mathrm{id},$$

$$e = G_1^*(e) - d_1\left\{\int_0^1 G_t^*(\chi \lrcorner e)\,dt\right\}, \qquad e \in E_1^{p,q}, \tag{9.11.1}$$

or identifying $E_1^{p,q}$ with $L_p(\kappa)$, we get

$$\Delta = G_1^*\{\Delta\} - \mathscr{E}_{p-1}\left[\int_0^1 G_t^*\{\chi_t \lrcorner \Delta\}\,dt\right], \qquad \Delta \in L_p(\kappa), \tag{9.11.2}$$

where $G_t^*\{\square\} = i^{-1}(G_t^*(i(\square)))$, $\square \in L_p(\kappa)$, and $i\colon L_p \to E_1^{p,q}$ is the identification map.

Formulas (9.11.1) and (9.11.2) remain obviously valid if the G_t are automorphisms of the object $\mathscr{Y}$ only for $t \in [0, 1)$. In particular, if $\operatorname{im} G_1 \subset \mathscr{Y}'$, where $\mathscr{Y}'$ is a subobject of the object $\mathscr{Y}$ such that $E_1^{p,q}(\mathscr{Y}') = 0$ these formulas give the necessary inversion.

If $\mathscr{Y} = N_m^\infty$ $(= J_{(\pi)}^\infty)$, then an important class of automorphisms of $\mathscr{Y}$ as an object of DE may be described in the following way. Suppose

$$G^\varepsilon\colon N_m^\varepsilon \to N_m^\varepsilon \qquad (G^\varepsilon\colon J^\varepsilon(\pi) \to J^\varepsilon(\pi))$$

is a diffeomorphism for $\varepsilon = 0$, $m > 1$ (or $\varepsilon = 0$, $\dim \pi > 1$) or a contact diffeomorphism for $\varepsilon = m = 1$ (or $\varepsilon = \dim \pi = 1$). Then G^{ε} may be naturally lifted to a diffeomorphisms $G^k \colon N_m^k \to N_m^k$ $(G^k \colon J^k(\pi) \to J^k(\pi))$, $\varepsilon \leqslant k \leqslant \infty$. Then $G = G^{\infty}$ is an automorphism in DE. A similar fact takes place for vector fields X^{ε} on $N_m^{\varepsilon}(J^{\varepsilon}(\pi))$ (see [4, 13]). Using formulas for lifting G^{ε} or X^{ε} (see [4, 13]), we can obtain explicit formulas for G^{∞} or X^{∞}. In particular, if $f = X^{\varepsilon} \lrcorner\, U_1(\pi)$, then the operator $\bar{X}_f \colon \mathscr{F}(\pi, \pi) \to \mathscr{F}(\pi, \pi)$ defined by the formula $\bar{X}_f(g) = \{f, g\}$ is of order 1 and moreover $\bar{X}_f \in \mathrm{Der}(\mathscr{F}(\pi, \pi))$ (see 1.5). Therefore the field $X_f \in D_{\mathscr{C}}(A)$ such that $\bar{X}(\varphi g) = X_f(\varphi) g + \varphi \bar{X}_f(g)$, where $\varphi \in A$, $g \in \mathscr{F}(\pi, \pi)$, is uniquely determined. It turns out that $X^{\infty} = X_f$. A field of the form X_f will further be called a Lie field (see [4, 13]). Thus, considering the families of transformations G_t^{ε} we can obtain the families $G_t = G_t^{\infty}$. Further we shall consider in great detail a practically important particular case.

Suppose $\pi \colon E_\pi \to M$ is a vector bundle and $G_t^0 \colon E_\pi \to E_\pi$ is its fiber-wise homotety with coefficients $1 - t$. Then $G_t = G_t^{\infty}$ is an automorphism for $t \in [0, 1)$ and $\operatorname{im} G_1 = \operatorname{im} j(0)$, $0 \in \Gamma(\pi)$. It follows from the latter that $G_1^*(E_1^{p,q}) = 0$, $p > 0$.

In standard coordinates (see 6.7) we have

$$G_t^*(x_k) = x_k, \qquad G_t^*(p_\sigma^i) = (1 - t) p_\sigma^i, \qquad t \in [0, 1],$$
$$X_t = \frac{1}{t-1} \sum_{i,\sigma} p_\sigma^i \frac{\partial}{\partial p_\sigma^i} = \frac{1}{t-1} 3_u, \qquad u = \rho_0(\pi). \tag{9.11.3}$$

Identifying according to 6.6 the elements of $\kappa(\pi)$ and of $\mathscr{F}(\pi, \pi)$ we can assume that $\chi_t = (1/(t-1))\, 3_u$ and, in view of (9.10.3), we have

$$(\chi_t \lrcorner\, \Delta)(f_1, \ldots, f_{p-1}) = \frac{1}{t-1} \Delta(u, f_1, \ldots, f_{p-1}),$$
$$f_i \in \mathscr{F}(\pi, \pi),\ \Delta \in L_{p+1}(\kappa). \tag{9.11.4}$$

LEMMA. *If $\Delta \in L_p(\kappa)$ then*

$$G_t\{\Delta\} = (1-t)^p\, (G_t^*)^{-1} \circ \Delta \circ G_t^* = (1-t)^p\, G_t(\Delta).$$

Proof. Since G_t is an automorphism of the bundle π, we have $G_t(3_f) = (G_t^*)^{-1} \circ 3_f \circ G_t^* = 3_{\tilde{f}}$ for some $\tilde{f}$ or $G_t^* \circ 3_{\tilde{f}} = 3_f \circ G_t^*$ (see [4]). Applying this relation to $u = \rho_0(\pi)$ and taking into consideration $3_g(u) = g$ and $G_t^*(u) = (1-t)u$ we find that $\tilde{f} = (1-t)[(G_t^*)^{-1}(f)]$. Thus

$$3_f \circ G_t^* = (1-t)\, G_t^* \circ 3_{(G_t^*)^{-1}(f)} \tag{9.11.5}$$

or equivalently,

$$l_{G_t^*(\varphi)} = (1-t)\, G_t^* \circ l_\varphi \circ (G_t^*)^{-1}. \tag{9.11.6}$$

Hence, in view of 1.5, we get

$$l^*_{G_t^*(\varphi)} = (1-t)\, G_t^* \circ l^*_\varphi \circ (G_t^*)^{-1}. \tag{9.11.7}$$

Since the A-module $L_p(\kappa)$ is generated by im $\mathscr{E}_{p-1}$ the necessary statement now follows from the fact that

$$\mathscr{E}_{p-1}((G_t^*)^{-1} \circ \Delta \circ G_t^*)) = (1-t)(G_t^*)^{-1} \circ \mathscr{E}_{p-1}(\Delta) \circ G_t^*, \qquad p > 1,$$

and $\mathscr{E}(G_t^*(h)) = (1-t)\, G_t^*(\mathscr{E}(h))$. The last relation means, in other words, that $l^*_{G_t^*(\omega)}(1) = (1-t)\, G_t^*(l^*_\omega(1))$, where $h = \langle \omega \rangle$, $\omega \in \bar{\Lambda}^n$, and immediately follows from (9.11.7). For $p > 1$ we must use (9.8.3), (9.11.5)–(9.11.7), and also the fact that

$$\{G_t^*(f), G_t^*(g)\} = (1-t)\, G_t^*(\{f, g\}).$$

But $\mathfrak{Z}_{G_t^*(f)}(G_t^*(g))$ in view of (9.11.5) equals $(1-t)\, G_t^*(\mathfrak{Z}_f(g))$ so that it remains to use (6.6.2). ∎

Putting together all the above for the family of homoteties from (9.11.4) we obtain for $\Delta \in L_p(\kappa)$, $\mathscr{E}_p(\Delta) = 0$, $p > 1$,

$$\Delta = \mathscr{E}_{p-1}\left(\int_0^1 [G_t^* \circ (\mathfrak{Z}_u \lrcorner \Delta) \circ (G_t^*)^{-1}](1-t)^{p-1}\, dt. \right. \tag{9.11.8}$$

The case $p = 1$ somewhat differs in form from the general one, since $G_t^*\{\alpha\} = G_t^*(\alpha)$, $\alpha \in \bar{H}^n$, and $\chi \lrcorner h = \langle h(\chi) \rangle \in \bar{H}^n$, where $h \in \hat{\kappa} = L_1(\kappa)$. Therefore in this case (9.11.2) yields

$$h = \mathscr{E}\left\langle \int_0^1 G_t^*(h(u)) \frac{dt}{1-t} \right\rangle, \qquad h(u) = h(\mathfrak{Z}_u). \tag{9.11.9}$$

9.12. *The "Potentials" of Trivial Lagrangians*

If Lagrangian $\mathscr{L} \in \bar{H}^n(B)$ is trivial and $H^n(B) = 0$ then, according to Proposition 9.9, we have $\omega = \bar{d}\rho$, where $\omega \in \bar{\Lambda}^n(B)$ and $\mathscr{L} = \langle \omega \rangle$. In this subsection we shall find an explicit formula for the "potential" ρ of the form ω. For every Lie field X_f we have the decomposition

$$X_f = \mathfrak{Z}_f + Y_f, \qquad Y_f \in \mathscr{C}D.$$

Therefore for $\omega \in \bar{\Lambda}^n$, by 7.2, we have

$$X_f(\omega) = \mathfrak{Z}_f(\omega) + \bar{d}(Y_f \lrcorner \omega) = \mathfrak{Z}_f(\omega) + \bar{d}(X_f \lrcorner \omega).$$

If $\mathscr{E}(\langle\omega\rangle) = l_\omega^*(1) = 0$ then by the Green $\mathscr{C}$-formula $\mathfrak{Z}_f(\omega) = l_\omega(f) = \bar{d}\,\text{Ж}_\lambda(l_\omega \circ f)$ so that in this case

$$X_f(\omega) = \bar{d}[\text{Ж}_\lambda(l_\omega \circ f) + X_f \lrcorner\, \omega].$$

If G_t is a family of automorphisms of domain B considered as an object of DE and $G_0 = \text{id}$, while $X_t = X_{f_t}$ is the corresponding family of $\mathscr{C}$-field, we have

$$G_1^*(\omega) - \omega = \int_0^1 \frac{d}{dt} G_t^*(\omega)\, dt = \int_0^1 G_t^*(X_t(\omega))\, dt$$
$$= \bar{d}\left[\int_0^1 G_t^*(\text{Ж}_\lambda(l_\omega \circ f_t) + X_{f_t} \lrcorner\, \omega)\, dt\right].$$

Thus, if $\mathscr{E}(\langle\omega\rangle) = 0$ we have

$$\omega = G_1^*(\omega) + \bar{d}\rho', \qquad \rho' = -\int_0^1 G_t^*(\text{Ж}_\lambda(l_\omega \circ f_t) + X_{f_t} \lrcorner\, \omega)\, dt. \quad (9.12.1)$$

If $G_1^*(\omega) = 0$ this formula gives the necessary expression for ρ. In particular, for families of homoteties, described in the previous section, since

$$G_1^*(\omega) = \pi_\infty^*(j(0)^*\,(\omega)) = \pi_\infty^*(\omega|_M), \qquad f_t = \frac{u}{t-1},$$

and $Y_{f_t} = 0$ we have

$$\omega = \pi_\infty^*(\omega|_M) + \bar{d}\rho', \qquad \rho' = \int_0^1 G_t^*\,\text{Ж}_\lambda(l_\omega \circ u)\frac{dt}{1-t}. \qquad (9.12.2)$$

Therefore under the condition

$$\int_M \omega|_M = 0 \Leftrightarrow \omega|_M = d\rho_0, \qquad \rho_0 \in \Lambda^{n-1}(M),$$

formula (9.12.2) gives the necessary formula for the potential ρ,

$$\rho = \rho' + \pi_\infty^*(\rho_0). \qquad (9.12.3)$$

10. The $\mathscr{C}$-Spectral Sequence of Infinitely Prolongated Equations

In this section we study the $\mathscr{C}$-spectral sequence of infinitely prolongated equations. We are mainly concerned with methods for describing the term

E_1, whose role in the category DE is entirely similar to the role of the de Rham complexes in the category of smooth manifolds. To do this we represent the complexes $E_0^{p,*}$ as the cokernels of a map of certain auxiliary complexes which are the poly-analogues of the Spencer complex. To compute these cokernels, it is necessary to construct in turn a different spectral sequence whose term E_0 is similar to the Spencer δ-complex. As a result it turns out to be possible to obtain a satisfactory description of the $\mathscr{C}$-spectral sequence for non-overdetermined equations. The methods of this section enable us to obtain useful estimates in the overdetermined case as well.

The "homotopy" theorem which is the analogue in DE of the "homotopy theorem" for de Rham cohomology, is proved at the end of this section. It enables to compute effectively in many cases the term E_2 of the $\mathscr{C}$-spectral sequence. In this work we only give its simplest corollary which is the analogue in DE of Poincare's lemma.

10.1 Suppose $\mathscr{Y} \subset N_m^k$ is an equation and $i = i_{\mathscr{Y}}\colon \mathscr{Y}_\infty \to N_m^\infty$ is inclusion. The $\mathscr{C}$-spectral sequence on N_m^∞ (resp. $\mathscr{Y}_\infty$) will be denoted by $\{E_r^{p,q}, d_r^{p,q}\}$ (resp. $\{E_r^{p,q}(\mathscr{Y}), d_r^{p,q}\}$). Since i is a morphism in DE, it induces a homomorphism of $\mathscr{C}$-spectral sequences, i.e., a system of homomorphisms $i^* = i_r^{p,q}\colon E_r^{p,q} \to E_r^{p,q}(\mathscr{Y})$ such that $i^* \circ d_r^{p,q} = d_r^{p,q} \circ i^*$. Obviously, $\operatorname{coker} i_0^{p,q} = 0$. To describe $\{E_r^{p,q}(\mathscr{Y}), d_r^{p,q}\}$ it is necessary, first of all to describe $\ker i_0^{p,q}$.

We shall first work within an affine chart or equivalently, we shall assume that $\mathscr{Y} \subset J^k(\pi)$ or $\mathscr{Y}_\infty \subset J^\infty(\pi)$. According to 9.2 we have $E_0^{p,q} = \Lambda^{p,q}$, $E_0^{p,q}(\mathscr{Y}) = \Lambda^{p,q}(\mathscr{Y})$ and the map $i_0^{p,q}\colon \Lambda^{p,q} \to \Lambda^{p,q}(\mathscr{Y})$ may be viewed as the natural restriction map on forms. The kernel of the restriction map $\Lambda^k \to \Lambda^k(\mathscr{Y}_\infty)$, as is shown in the general theory of differential forms, coincides with $\mathscr{I}_{\mathscr{Y}} \cdot \Lambda^k + \Lambda^{k-1} \wedge d\mathscr{I}_{\mathscr{Y}}$, where $\mathscr{I}_{\mathscr{Y}}$ is the ideal of the equation $\mathscr{Y}_\infty \subset J^\infty(\pi)$. Taking the (p, q)-component of this kernel we see that

$$\ker i_0^{p,q} = \mathscr{I}_{\mathscr{Y}} \cdot \Lambda^{p,q} + \mathscr{F} \cdot U_1(\mathscr{I}_{\mathscr{Y}}) \wedge \underbrace{\mathscr{C}\Lambda^1 \wedge \cdots \wedge \mathscr{C}\Lambda^1}_{p-1 \text{ times}} \wedge \Lambda_0^q$$

$$+ \underbrace{\mathscr{C}\Lambda^1 \wedge \cdots \wedge \mathscr{C}\Lambda^1}_{p \text{ times}} \wedge \hat{d}\mathscr{I}_{\mathscr{Y}} \wedge \Lambda_0^{q-1}, \qquad \mathscr{F} = \mathscr{F}(\pi).$$

But obviously, $\hat{d}\mathscr{I}_{\mathscr{Y}} \subset \mathscr{I}_{\mathscr{Y}} \cdot \Lambda^1$. Therefore $\hat{d}\mathscr{I}_{\mathscr{Y}} \subset \mathscr{I}_{\mathscr{Y}}\Lambda_0^1$ and

$$\ker i_0^{p,q} = \mathscr{I}_{\mathscr{Y}} \cdot \Lambda^{p,q} + \mathscr{F} \cdot U_1(\mathscr{I}_{\mathscr{Y}}) \wedge \Lambda^{p-1,q}.$$

In particular $\ker i_0^{1,0} = \mathscr{I}_{\mathscr{Y}} \cdot \mathscr{C}\Lambda^1 + \mathscr{F} \cdot U_1(\mathscr{I}_{\mathscr{Y}})$. It follows from this equality that, despite the fact that the operator U_1 depends on the choice of the affine chart, the image of the submodule $\mathscr{F} \cdot U_1(\mathscr{I}_{\mathscr{Y}})$ in $\mathscr{C}\Lambda^1/\mathscr{I}_{\mathscr{Y}} \cdot \mathscr{C}\Lambda^1$

is well defined. Thus, if we denote by $[P]=P/\mathscr{I}_{\mathscr{Y}}\cdot P$ the restriction of the $\mathscr{F}$-module P to $\mathscr{Y}_\infty$ we obtain

$$[\ker i_0^{p,q}]=[\mathscr{F}\cdot U_1(\mathscr{I}_{\mathscr{Y}})]\wedge[E_0^{p-1,q}], \qquad (10.1.1)$$

where $[\mathscr{F}\cdot U_1(\mathscr{I}_{\mathscr{Y}})]$ denotes the submodule in $[\mathscr{C}\Lambda^1]$ generated within a certain affine chart by the set $U_1(\mathscr{I}_{\mathscr{Y}})$ while the oerator U_1 is taken within this chart. Note that $[\mathscr{F}\cdot U_1(\mathscr{I}_{\mathscr{Y}})]=[\ker i_0^{1,0}]$.

10.2. Now we would like to obtain the description of the term $E_0^{p,q}(\mathscr{Y})$ similar to the description of the term $E_0^{p,q}$ obtained in 9.2. To do this, to each $\varphi\in[E_0^{p,q}]$ we assign the operator

$$\nabla_\varphi\in\mathscr{C}\operatorname{Diff}^{\text{alt}}_{*,p}([\kappa];\bar{\Lambda}^q(\mathscr{Y}))$$

by putting

$$\nabla_\varphi([\chi_1],\dots,[\chi_p])=[\chi_p\,\lrcorner\,(\cdots(\chi_1\,\lrcorner\,\rho)\cdots)], \qquad (10.2.1)$$

where $\chi_i\in\kappa$ and $[\chi_i]$ is the image of χ_i in $[\kappa]$, while $\varphi=[\rho]$. It should be noted also that the right-hand side of (10.2.1) belongs to $[E_0^{0,q}]=[\bar{\Lambda}^q]$. However, obviously, $[\bar{\Lambda}^q]=\bar{\Lambda}^q(\mathscr{Y})$. Word for word as in Proposition 2.7 we can prove the following:

PROPOSITION. *The correspondence $\varphi\mapsto\nabla_\varphi$ establishes an isomorphism of the $\mathscr{F}(\mathscr{Y})$-modules $[E_0^{p,q}]$ and $\mathscr{C}\operatorname{Diff}^{\text{alt}}_{*,p}([\kappa];\bar{\Lambda}^q(\mathscr{Y}))$.*

Note that if $\rho\in E_0^{p,q}$ then $\nabla_{[\rho]}=[\nabla_\rho]$, where $[\Delta]$ denotes the restriction of the $\mathscr{C}$-polydifferential operator Δ from N_m^∞ (or $J^\infty(\pi)$) to $\mathscr{Y}_\infty$ (see 6.4).

The bigraded algebra

$$\mathscr{C}\operatorname{Diff}^{\text{alt}}\{[\kappa]\}=\sum_{p,q}\mathscr{C}\operatorname{Diff}^{\text{alt}}_{*,p}([\kappa];\bar{\Lambda}^q(\mathscr{Y}))$$

over $\mathscr{F}(\mathscr{Y})$, unlike $\mathscr{C}\operatorname{Diff}^{\text{alt}}\{\kappa\}$ is no longer a bicomplex. The differentials

$$\bar{d}=\bar{d}_{p,q}\colon\mathscr{C}\operatorname{Diff}^{\text{alt}}_{*,p}([\kappa];\bar{\Lambda}^q(\mathscr{Y}))\to\mathscr{C}\operatorname{Diff}^{\text{alt}}_{*,p}([\kappa]);\bar{\Lambda}^{q+1}(\mathscr{Y}))$$

are introduced into it by means of the formula (9.4.2). However, formula (9.4.1), which introduces the differential $l_{p,q}$, no longer has any meaning in the situation at hand.

10.3. Consider the $\mathscr{F}(\mathscr{Y})$-submodules

$$K_0^{p,q}=\{\nabla_\varphi\mid\varphi\in[\ker i_0^{p,q}]\}\subset\mathscr{C}\operatorname{Diff}^{\text{alt}}_{*,p}([\kappa];\bar{\Lambda}^q(\mathscr{Y})),$$
$$K_0=\sum_{p,q}K_0^{p,q}\subset\mathscr{C}\operatorname{Diff}^{\text{alt}}\{[\kappa]\}.$$

It follows from (10.1.1) and Proposition 10.2 that K_0 is a bigraded ideal of the algebra $\mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}\{[\kappa]\}$ generated by $K_0^{1,0}$, i.e.,

$$K_0^{p,q} = K_0^{1,0} \wedge \mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{*,p-1}([\kappa]; \bar{\Lambda}^q(\mathscr{Y})).$$

Obviously, $\bar{d}_{1,0}(K_0^{1,0}) \subset K_0^{1,0}$. Together with (10.3.1) and (9.4.3) this shows that $\bar{d}(K_0) \subset K_0$.

Consider the bigraded quotient algebra

$$K = \mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}\{[\kappa]\}/K_0 = \sum_{p,q} K^{p,q},$$

$$K^{p,q} = \mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{*,p}([\kappa]; \bar{\Lambda}^q)/K_0^{p,q}.$$

In view of the above, this quotient algebra inherits the differentials $\bar{d}_{p,q}$. The corresponding quotient differentials still will be denoted by $\bar{d}_{p,q}$,

$$\bar{d} = \bar{d}_{p,q}\colon K^{p,q} \to K^{p,q+1}.$$

The isomorphism of Proposition 10.2, in view of the fact that $E_0^{p,q}(\mathscr{Y}) = [E_0^{p,q}]/[\ker i_0^{p,q}]$ induces an isomorphisms

$$И = И^{p,q}\colon K^{p,q} \to E_0^{p,q}(\mathscr{Y}), \qquad И\colon K \to E_0(\mathscr{Y})$$

while Proposition 9.4 shows that this isomorphism identifies the differentials $\bar{d}_{p,q}$ with the differentials $d_0^{p,q}(\mathscr{Y})$.

If $\mathscr{Y}_\infty \subset J^\infty(\pi)$ then, according to construction 9.2, $E_0^{p,q}(\mathscr{Y})$ is identified with $\Lambda^{p,q}(\mathscr{Y})$. Therefore, by using the isomorphism $И^{p,q}$ we can identify $K^{p,q}$ with $\Lambda^{p,q}(\mathscr{Y})$ and K with $\Lambda^*(\mathscr{Y}_\infty)$. Then the differentials $\bar{d}_{p,q}$ are identified with $d'_{p,q}$. Let us describe the differentials $K^{p,q} \to K^{p+1,q}$ which, under the considering identifications, are mapped into $d''_{p,q}$.

First note that if $\omega \in \bar{\Lambda}^q$, $X \in D_{\mathscr{C}}$ and $X = 0$ on $\mathscr{Y}_\infty$ then $X(\omega) = 0$ on $\mathscr{Y}_\infty$. Indeed, if $X = \mathfrak{Z}_\varphi$ then $\mathfrak{Z}_\varphi(\omega) = l_\omega(\varphi)$ and in view of the fact that the ideal $\mathscr{I}_{\mathscr{Y}}$ is deifferentially closed, $l_\omega(\varphi) = 0$ on $\mathscr{Y}_\infty$. If $X \in \mathscr{C}D$ then $X(\omega) = \bar{d}(X \rfloor \omega) + X \rfloor \bar{d}\omega$. But $(X \rfloor \omega)|_{\mathscr{Y}_\infty} = 0$ and $(X \rfloor \bar{d}\omega)|_{\mathscr{Y}_\infty} = 0$ if $X = 0$ on $\mathscr{Y}_\infty$, while $\bar{d}(X \rfloor \omega)|_{\mathscr{Y}_\infty} = 0$ in view of the fact that the ideal $\mathscr{I}_{\mathscr{Y}}$ is differentiably closed.

It follows from the above that the action of the $\mathscr{F}(\mathscr{Y})$-module $[\kappa]$ on the $\mathscr{F}$-modules $\bar{\Lambda}^q$ with values in $[\bar{\Lambda}^q]$ defined according to the formula

$$([\chi], \omega) \mapsto [\chi(\omega)], \qquad \chi \in \kappa,\ \omega \in \bar{\Lambda}^q,$$

is defined correctly. Then the operator $\square \in \mathscr{C}\,\mathrm{Diff}([\kappa], \bar{\Lambda}^q)$, $\square([\chi]) = [\chi(\omega)]$ in view of the fact that $\chi(\omega) = \chi \rfloor U_1(\omega)$ belongs to the ideal K_0, whenever $\omega|_{\mathscr{Y}_\infty} = 0$.

The fact that the ideal $\mathscr{I}_{\mathscr{Y}}$ is differentiably closed, shows that $\{\varphi, \psi\} =$

$l_\psi(\varphi) - l_\varphi(\psi) = 0$ on $\mathscr{Y}_\infty$ if $\varphi|_{\mathscr{Y}_\infty} = \psi|_{\mathscr{Y}_\infty} = 0$ and that $\Delta(\chi_1, \ldots, \chi_p) = 0$ on $\mathscr{Y}_\infty$ whenever Δ is a $\mathscr{C}$-differential operator and $\chi_i|_{\mathscr{Y}_\infty} = 0$.

All this taken together shows that formula (9.4.1) induces operators $l = l_{p,q}$ in K. Thus we have proved the following:

PROPOSITION. *The algebra K supplied with the differential $\bar{d}$, is isomorphic to $\{E_0(\mathscr{Y}), d_0\}$. In the case $\mathscr{Y}_\infty \subset J^\infty(\pi)$, the algebra K with the differentials $\bar{d}$ and l becomes a bicomplex isomorphic to the bicomplex $\{\Lambda^*(\mathscr{Y}), d', d''\}$.*

Remark. If $\mathscr{Y}_\infty \subset J^\infty(\pi)$, then the algebra K with the differential l is isomorphic to the term E_0 of the de Rham cohomology spectral sequence of the map $\mathscr{Y}_\infty \to M$ (see Remark 2 in 9.2).

10.4 In the "regular" situation, which is described further the complexes

$$K^{p,*} = \sum_q K^{p,q} \subset K$$

(with respect to the differential $\bar{d}$) may be represented in the form of the cokernel of a map of complexes of the Spencer type. This gives an approach to the calculation of the terms $E_1^{p,q}(\mathscr{Y})$ since $H^q(K^{p,*}) = E_1^{p,q}(\mathscr{Y})$.

The equation $\mathscr{Y} \subset J^k(\pi)$ will be called regular if $\mathscr{Y} = \{\varphi = 0\}$, where $\varphi \in \mathscr{F}_k(\pi, \xi)$ for some $\xi \colon E_\xi \to M$ and

$$\mathscr{I}_{\mathscr{Y}} = \mathscr{C}\,\mathrm{Diff}(P, \mathscr{F})(\varphi),$$

where $P = \mathscr{F}(\pi, \xi)$. For a regular equation the following simple but important lemma is true and yields the necessary interpretation of the complexes $K^{p,*}$.

LEMMA. *If $\mathscr{Y}$ is regular then*

$$K_0^{1,0} = \square \in \mathscr{C}\,\mathrm{Diff}([\kappa], \mathscr{F}(\mathscr{Y})) \mid \square = \Delta \circ [l_\varphi],\ \Delta \in \mathscr{C}\,\mathrm{Diff}([P], \mathscr{F}(\mathscr{Y}))\}$$

(where the brackets $[\cdot]$ denote the restriction of a $\mathscr{C}$-differential operator from $J^\infty(\pi)$ to $\mathscr{Y}_\infty$).

Proof. According to 10.1, $K_0^{1,0}$ viewed as a $\mathscr{F}(\mathscr{Y})$-module is generated by the operators $[l_f]$, $f \in \mathscr{I}_{\mathscr{Y}}$. In view of the fact that the equation $\mathscr{Y}$ is regular we have $f = \nabla(\varphi)$, $\nabla \in \mathscr{C}\,\mathrm{Diff}(P, \mathscr{F})$ and according to 7.1 $\nabla = \sum_i a_i \hat{\nabla}_i$, $a_i \in \mathscr{F}$. But $l_{\hat{\square}(\psi)} = \hat{\square} \circ l_\psi$ so that taking into consideration the fact that the operation $g \mapsto l_g$ is a derivation, we obtain

$$l_f = \sum_i a_i \hat{\nabla}_i \circ l_\varphi + \sum_i \hat{\nabla}_i(\varphi)\, l_{a_i}.$$

Since $\hat{\nabla}_i(\varphi) \in \mathscr{I}_{\mathscr{Y}}$, we have $[l_\varphi] = \sum [a_i \hat{\nabla}_i] \circ [l_\varphi]$. ∎

In view of the projectivity of the $\mathscr{F}(\mathscr{Y})$-modules $[P]$ and $[\kappa]$ we have

$$\mathscr{C}\operatorname{Diff}([P], \bar{\Lambda}^q(\mathscr{Y})) = \mathscr{C}\operatorname{Diff}([P], \mathscr{F}(\mathscr{Y})) \otimes_{\mathscr{F}(\mathscr{Y})} \bar{\Lambda}^q(\mathscr{Y})$$

and similarly for $[\kappa]$. This and the lemma above imply

$$K_0^{1,q} = \{\square \in \mathscr{C}\operatorname{Diff}([\kappa], \bar{\Lambda}^q(\mathscr{Y})) \mid \square = \Delta \circ [l_\varphi],$$
$$\Delta \in \mathscr{C}\operatorname{Diff}([P], \bar{\Lambda}^q(\mathscr{Y}))\}.$$

This means that the complex $K_0^{1,*} = \sum_q K_0^{1,q}$ is the image of the map

$$\mathscr{S}_{[l_\varphi]}\colon \widetilde{\mathscr{S}[P]} \to \widetilde{\mathscr{S}[\kappa]}$$

(see (4.2) so that

$$K^{1,*} = \operatorname{coker} \mathscr{S}_{[l_\varphi]}$$

10.5. Now suppose $\mathscr{Y} \subset N_m^k$ and $\mathscr{Y} = \{\varphi = 0\}$, $\varphi \in P$. The definition of the regularity of an equation given in the previous subsection can be carried over word for word in the case considered. However, Lemma 10.4 no longer makes sense, since the operator l_φ is not defined. Let us construct its substitutes. To do this we introduce the following terminilogy:

DEFINITION. (1) The equation $\mathscr{Y} \subset N_m^k$ will be called locally regular, if there exists the covering $\{\mathscr{U}_\alpha\}$ of the space N_m^∞ by affine charts such that the equation $\mathscr{Y}$ is regular within any one of them.

(2) The equation $\mathscr{Y} \subset N_m^k$ will be called regular with respect to the operator $\nabla \in \mathscr{C}\operatorname{Diff}([\kappa], [P])$ if

$$K_0^{1,0} = \{\square \in \mathscr{C}\operatorname{Diff}([\kappa], \mathscr{F}(\mathscr{Y})) \mid \square = \Delta \circ \nabla,$$
$$\Delta \in \mathscr{C}\operatorname{Diff}([P], \mathscr{F}(\mathscr{Y}))\}.$$

Remark. It follows from Lemma 10.4 that a regular equation $\mathscr{Y} = \{\varphi = 0\} \subset J^k(\pi)$ is regular with respect to the operator $[l_\varphi]$.

PROPOSITION. (1) *If the equation $\mathscr{Y}$ is regular with respect to the operator ∇ then* $K^{1,*} = \operatorname{coker} \mathscr{S}_\nabla$.

(2) *If the equation $\mathscr{Y}$ is locally regular, then there exists an operator ∇ with respect to which it is regular.*

Proof. The first statement is obvious. To prove the second, we shall give a formula for the transformation of the operator l_φ under a change of affine charts. This formula is an immediate consequence of the transformational

properties of the operation U_1 (see [4, 13]) and the relationship of this operation with the operation l (see (6.6). Thus we have

$$\tilde{l}_\varphi = l_\varphi + \sum_{i,k} D_k(\varphi_i)\, \tilde{l}_{x_i \psi_k} + \sum_i \varphi_i \tilde{l}_{\psi_i},$$

where (x^i, p^j_σ) are coordinates in the first chart, ψ_i is the basis of the module P in this same chart, $D_k = \hat{\partial}/\partial x_k$, and $\tilde{l}$ is the operation of universal linearization in the sense of the second chart. This formula immediately implies that $[\tilde{l}_\varphi] = [l_\varphi]$ (in the intersection of the two charts). Therefore an operator ∇ such that its restriction to any local chart is of the form $[l_\varphi]$, where the operation l is viewed in the sense of this chart, is correctly defined. Further, standard partition of unity techniques and Lemma 10.4 show that the equation $\mathscr{Y}$ is regular with respect to the operator ∇ thus constructed. ∎

10.6 The representation of the complex $K^{1,*}$ in the form of the cokernel of the map $\mathscr{S}_\nabla$ is the key, since it reduces the computation of the term $E_1^{1,q}$ to the computation of the Spencer cohomology of the operator ∇. The latter in many cases turns out to be an effective procedure. For this reason it is desirable to have a similar representations for the complexes $K^{p,*}$, $p > 1$, as well. It is defined in this section.

Suppose

$$\mathscr{C}\,\mathrm{Diff}_{(k)}(Q, Q'; \bar{\Lambda}^q(\mathscr{Y})) = \mathscr{C}\,\mathrm{Diff}(\underbrace{Q,\dots, Q}_{k\,\text{times}}, Q'; \bar{\Lambda}^q(\mathscr{Y})),$$

$$\mathscr{C}\,\mathrm{Diff}_{(k)}\{Q; Q'\} = \sum_q \mathscr{C}\,\mathrm{Diff}_{(k)}(Q, Q'; \bar{\Lambda}^q(\mathscr{Y})),$$

where Q and Q' are some $\mathscr{F}(\mathscr{Y})$-modules. According to 9.5 $\mathscr{C}\,\mathrm{Diff}_{(k)}\{Q, Q'\}$ is a complex with respect to the differential $\bar{d}$ in which the permutation group S_k acts naturally by interchanging the first k "arguments." The following subcomplex is invariant with respect to this action

$$\mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{(k)}\{Q; Q'\} = \sum_q \mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{(k)}(Q, Q'; \bar{\Lambda}^q(\mathscr{Y})).$$

Note the following isomorphisms which shall be used further without special mention:

$$\mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{(k)}(Q, Q'; \bar{\Lambda}^q(\mathscr{Y})) = \mathscr{C}\,\mathrm{Diff}(Q', \bar{\Lambda}^q(\mathscr{Y})) \otimes_{\mathscr{F}(\mathscr{Y})} \mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{*,k}(Q; \mathscr{F}(\mathscr{Y})), \qquad (10.6.1)$$

$$\mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{(k)}\{Q; Q'\} = \widetilde{\mathscr{S}Q'} \otimes_{\mathscr{F}(\mathscr{Y})} \mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{*,k}(Q; \mathscr{F}(\mathscr{Y})).$$

About $\widetilde{\mathscr{S}Q'}$ see 4.2. It goes without saying that in (10.6.1) the superscript alt may be omitted.

Now consider the composition $v_\nabla^{(p)}$ of the maps

$$\begin{aligned}&\widetilde{\mathscr{S}[P]}\otimes_{\mathscr{F}(\mathscr{Y})}\mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{*,p-1}([\kappa]);\mathscr{F}(\mathscr{Y}))\\ &\xrightarrow{\mathscr{S}_\nabla\otimes\mathrm{id}} K_0^{1,*}\otimes_{\mathscr{F}(\mathscr{Y})}\mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{*,p-1}([\kappa];\mathscr{F}(\mathscr{Y}))\\ &\longrightarrow K_0^{1,*}\wedge\mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{*,p-1}([\kappa];\mathscr{F}(\mathscr{Y}))=K_0^{p,*}.\end{aligned}$$

Concerning the maps $\mathscr{S}_\nabla$ see 4.2. Also recall that according to 10.4 and 10.5 we have $K_0^{1,*}=\mathrm{im}\,\mathscr{S}_\nabla\subset\widetilde{\mathscr{S}_{[\kappa]}}$. Obviously $v_\nabla^{(p)}$ is an epimorphism.

Thus we have the following:

PROPOSITION. *If the equation $\mathscr{Y}$ is regular with respect to ∇ then $K^{p,*}=$ coker $v_\nabla^{(p)}$, $p\geqslant 1$.*

It is useful to note that the composition of $v_\nabla^{(p)}$ and the inclusion $K_0^{p,*}\subset\mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{(p-1)}\{[\kappa]\}$ coincides with the composition

$$\begin{aligned}&\mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{(p-1)}\{[\kappa];[P]\}\\ &\xrightarrow{\mathscr{C}\,\mathrm{Diff}(\mathrm{id},\nabla)}\mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{(p-1)}\{[\kappa];[\kappa]\}\xrightarrow{\mathrm{alt}}\mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{(p)}\{[\kappa]\}.\end{aligned}\tag{10.6.2}$$

10.7. We now pass to the study of the cohomology of the complexes $K^{p,*}$, $p\geqslant 1$. First recall (see 9.5) that the complex $\mathscr{C}\,\mathrm{Diff}_{(k)}\{Q;Q'\}$ is acyclic in dimensions $i\neq n$ and

$$H^n(\mathscr{C}\,\mathrm{Diff}_{(k)}\{Q;Q'\})=\mathscr{C}\,\mathrm{Diff}_{*,k}(Q;\hat{Q}').$$

For this reason, the complex $\mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{(k)}\{Q;Q'\}$, being invariant with respect to the action of the permutation group S_k in $\mathscr{C}\,\mathrm{Diff}_{(k)}\{Q;Q'\}$ and, being a direct summand, is also acyclic in dimensions $i\neq n$ and

$$H^n(\mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{(k)}\{Q;Q'\})'=\mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{*,k}(Q;\hat{Q}').\tag{10.7.1}$$

Consider the following short exact sequences of complexes

$$\begin{tikzcd}0\arrow[r] & \ker v_\nabla^{(p)}\arrow[r] & \mathscr{S}[P]\otimes_{\mathscr{F}(\mathscr{Y})}\mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{*,p-1}([\kappa];\mathscr{F}(\mathscr{Y}))\arrow[r, "v_\nabla^{(p)}"]\arrow[d, leftrightarrow, "\approx"] & K_0^{p,*}\arrow[r] & 0\\ & & \mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{(p-1)}\{[\kappa];[P]\} & & \end{tikzcd}$$

$$0\to K_0^{p,*}\to\mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{(p)}\{[\kappa]\}\to K^{p,*}\to 0.$$

From this corresponding cohomology exact sequences, we find, by using 9.5. and (10.7.1), that

$$H^i(K^{p,*}) = H^{i+1}(K_0^{p,*}) = H^{i+2}(\ker v_\nabla^{(p)}), \qquad i < n-2,$$
$$H^{n-2}(K^{p,*}) = H^{n-1}(K_0^{p,*}), \tag{10.7.2}$$

$$0 \to H^{n-1}(K_0^{p,*}) \to H^n(\ker v_\nabla^{(p)}) \xrightarrow{I_p} \mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{*,p-1}([\kappa]; [\hat{P}])$$
$$\to H^n(K_0^{p,*}) \to 0,$$
$$0 \to H^{n-1}(K^{p,*}) \to H^n(K_0^{p,*}) \to L_p([\kappa]) \to H^n(K^{p,*}) \to 0.$$

Recall that the cohomology of the complexes considered here is trivial in dimensions $i < 0$ and $i > n$.

It follows from (10.7.2) that the computation of the cohomology of the complex $K^{p,*}$ reduces in principle to the computation of the cohomology of the complex $\ker v_\nabla^{(p)}$. In particular, if this complex is acyclic in dimensions $i \neq n$ and the map

$$I_p\colon H^n(\ker v_\nabla^{(p)}) \to \mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{*,p-1}([\kappa]; [\hat{P}])$$

is a monomorphism, then $H^i(K^{p,*}) = 0$, $i \neq n-1, n$. Moreover, we always have

$$H^n(K^{p,*}) = \operatorname{coker} \nabla_{(p)},$$

where $\nabla_{(p)}\colon \mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{*,p-1}([\kappa]; [\hat{P}]) \to L_p([\kappa])$ can be found from the following diagram of the map of n-dimensional cohomology of the complex $\mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{(p-1)}\{[\kappa]; [P]\}$ into $H^n(\mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{(p)}([\kappa]) = L_p([\kappa])$, corresponding to the map $v_\nabla^{(p)}$,

$$\begin{array}{ccc}
\mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{p-1}([\kappa], [P]; \bar{\Lambda}^n(\mathscr{Y})) & \xrightarrow{v_\nabla^{(p)}} & \mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{*,p}([\kappa]; \bar{\Lambda}^n(\mathscr{Y})) \\
\downarrow & & \downarrow \\
\mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{p-1}([\kappa], [P]; \bar{\Lambda}^n(\mathscr{Y}))/\mathrm{im}\,\bar{d} & \xrightarrow{\nabla_{(p)}} & \mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{*,p}([\kappa]; \bar{\Lambda}^n(\mathscr{Y}))/\mathrm{im}\,\bar{d} \\
\updownarrow\approx & & \updownarrow\approx \\
\mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{*,p-1}([\kappa]; [\hat{P}]) & \xrightarrow{\nabla_{(p)}} & L_p([\kappa]).
\end{array}$$

It immediately follows from definitions and from 9.5 that $\nabla_{(p)}$ is the composition of maps

$$\mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{*,p-1}([\kappa]; [\hat{P}]) \xrightarrow{\alpha_p} \mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{*,p-1}([\kappa]; [\hat{\kappa}]) \xrightarrow{\beta_p} L_p([\kappa]),$$

where identifying $\mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{*,p-1}([\kappa];[\hat{P}])$ with the skew-symmetric $[\hat{P}]$-valued polydifferential operators, we have

$$\alpha_p(\Delta)(\chi_1,\dots,\chi_{p-1})=\nabla^*(\Delta(\chi_1,\dots,\chi_{p-1})),\qquad \chi_i\in[\kappa],$$

where the map β_p is the alternation map with respect to the action of the permutation group S_p in $\mathscr{C}\,\mathrm{Diff}_{*,p-1}([\kappa];[\hat{\kappa}])$ described in 9.6. Note that $\nabla_{(1)}=\nabla^*$.

Since $\nabla_{(p)}$ is the composition of the maps (see (10.7.2))

$$\mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}_{*,p-1}([\kappa];[\hat{P}])\to H^n(K_0^{p,*})\to L_p([\kappa]),$$

we have the exact sequence

$$0\to H^{n-1}(K_0^{p,*})\to H^n(\ker v_\nabla^{(p)})\xrightarrow{I_p}\ker\nabla_{(p)}\to H^{n-1}(K^{p,*})\to 0.\quad (10.7.3)$$

Bringing together the above and using Proposition 10.3 we obtain the following result:

THEOREM (The two-line theorem). *Suppose equation $\mathscr{Y}$ is regular with respect to the operator ∇ and* (a) *the complexes $K_0^{p,*}$ for $p\geqslant 1$ are acyclic in dimensions $i\neq n$ or, equivalently,* (b) *the complexes* $\ker v_\nabla^{(p)}$ *for $p\geqslant 1$ are acyclic in dimensions $i\neq n$ and I_p is a monomorphism. Then*

(1) $E_1^{p,q}(\mathscr{Y})=0$ *if* $p\geqslant 1$, $q\neq n-1$, n;

(2) $E_1^{p,n}(\mathscr{Y})=\operatorname{coker}\nabla_{(p)}$

(3) *the following sequence is exact*

$$0\to H^n(\ker v_\nabla^{(p)})\to\ker\nabla_{(p)}\to E_1^{p,n-1}(\mathscr{Y})\to 0.$$

Recall that $E_1^{0,q}(\mathscr{Y})=\bar{H}^q(\mathscr{Y})$ (Proposition 1.9.1). Therefore, under our assumptions, the nonzero terms $E_1^{p,q}(\mathscr{Y})$ (if we discount the "segment" $0\leqslant q\leqslant n$ of the column $p=0$) are located in the two lines $q=n-1$ and $q=n$, forming the complexes

$$0\to\bar{H}^{n-1}(\mathscr{Y})\xrightarrow{d_1^{0,n-1}}E_1^{1,n-1}\xrightarrow{d_1^{1,n-1}}\cdots\to E_1^{p,n-1}\xrightarrow{d_1^{p,n-1}}\cdots,\quad (10.7.4)$$

$$0\to\bar{H}^n(\mathscr{Y})\xrightarrow{d_1^{0,n}}E_1^{1,n}=\operatorname{coker}\nabla^*\xrightarrow{d_1^{1,n}}\cdots\to\operatorname{coker}\nabla_{(p)}\xrightarrow{d_1^{p,n}}\cdots.\quad (10.7.5)$$

The following is a consequence of the "two line theorem":

COROLLARY. *Under the assumptions of the two-line theorem, we have*

(1) $E_r^{p,q}(\mathscr{Y})=0$ *if* $p\geqslant 1$, $1\leqslant r\leqslant\infty$ *and* $q\neq n-1$ *or* n;

(2) $E_3^{p,q}(\mathscr{Y})=E_\infty^{p,q}$;

(3) $\bar{H}^q(\mathscr{Y}) = E_1^{0,q}(\mathscr{Y}) = E_\infty^{0,q}(\mathscr{Y}) = H^q(\mathscr{Y}_\infty)$ *if* $q \leqslant n-2$;

(4) $E_2^{0,n-1}(\mathscr{Y}) = E_\infty^{0,n-1}(\mathscr{Y}) = H^{n-1}(\mathscr{Y}_\infty), E_2^{1,n-1}(\mathscr{Y}) = E_\infty^{1,n-1}(\mathscr{Y})$. ∎

As we see from this corollary, for the computation of the $\mathscr{C}$-spectral sequence we must know the groups $H^q(\mathscr{Y}_\infty)$. Their description does not lead to any difficulty as a rule; in this connection see 11.1.

The significance of the "two-line theorem" is due to the fact that its assumptions hold, as a rule, for non-overdetermined equations (see below). Thus it describes the structure of the $\mathscr{C}$-spectral sequence in this case.

Remark. It is convenient to use the assumptions of the two-line theorem in the form (b) in practice in cases when the kernel $\ker v_\nabla^{(p)}$ is "small", i.e., when the equation $\mathscr{Y}$ is not "very overdetermined." Further, in our general arguments, we will use its assumptions in the form (a).

10.8. Let us show first of all that non-overdetermined equations almost always satisfy $\ker v_\nabla^{(1,q)} = 0$. Since

$$v_\nabla^{(1,q)} = \mathscr{S}_\nabla|_{\mathscr{C}\,\mathrm{Diff}([P],\bar{\Lambda}^q(\mathscr{Y}))},$$

we must find conditions under which $\ker \mathscr{S}_\nabla = 0$. To do this, we will use, in the $\mathscr{C}$-theory on $\mathscr{Y}_\infty$, the same arguments as in 4.3.

If $\mathscr{Y}_\infty \subset J^\infty(\pi)$, then the maximal spectrum of the algebra $\mathscr{C}\,\mathrm{smbl}\,\mathscr{F}(\mathscr{Y})$ (see 7.4.) contains the total space of the bundle $T^*(M, \mathscr{Y}_\infty) \to \mathscr{Y}_\infty$ induced by the projection $\mathscr{Y}_\infty \to M$ from the cotangent bundle $T^*(M) \to M$ and every element from $\mathscr{C}\,\mathrm{smbl}\,\mathscr{F}(\mathscr{Y})$ may be viewed as a function on $T^*(M, \mathscr{Y}_\infty)$.

Suppose $\xi_i\colon E_\xi \to M$, $i = 1, 2$, are linear bundles, $Q_i = \mathscr{F}(\pi, \xi_i)$ and $\tilde{\xi}_i\colon \tilde{E}_i \to T^*(M, \mathscr{Y}_\infty)$ is the bundle induced by the natural projection $T^*(M, \mathscr{Y}_\infty) \to M$ from the bundle ξ_i. Then since

$$\mathscr{C}\,\mathrm{smbl}(Q_1, Q_2) = \mathrm{Hom}_{\mathscr{F}(\mathscr{Y})}(Q_1, \mathscr{C}\,\mathrm{smbl}\,Q_2)$$

(see 7.4.), every element $s \in \mathscr{C}\,\mathrm{smbl}(Q_1, Q_2)$ may be interpreted as a homomorphism $h(s)\colon \tilde{\xi}_1 \to \tilde{\xi}_2$.

Recall that the module $\mathscr{C}\,\mathrm{smbl}(Q_1, Q_2)$ has the grading associated with the filtered module

$$\mathscr{C}\,\mathrm{Diff}(Q_1, Q_2) = \{\mathscr{C}\,\mathrm{Diff}_i(Q_1, Q_2)\}_{i \geqslant 0}.$$

Hence the homomorphism

$$\mathscr{S}_\nabla\colon \mathscr{C}\,\mathrm{Diff}([P], \bar{\Lambda}^q(\mathscr{Y})) \to \mathscr{C}\,\mathrm{Diff}([\kappa], \bar{\Lambda}_q(\mathscr{Y}))$$

generates the homomorphism

$$\mathscr{S}_\nabla^{gr}: \mathscr{C}\,\mathrm{smbl}([P], \bar{\Lambda}^q(\mathscr{Y})) \to \mathscr{C}\,\mathrm{smbl}([\kappa], \bar{\Lambda}^q(\mathscr{Y})),$$
$$\mathscr{S}_\nabla^{gr}(\mathscr{C}\,\mathrm{smbl}_i\,\square) = \mathscr{C}\,\mathrm{smbl}_{i+k}(\square \circ \nabla),\ k = \mathrm{ord}\,\nabla.$$

Obviously $\ker \mathscr{S}_\nabla^{gr} = 0$ implies $\ker \mathscr{S}_\nabla = 0$. Hence it suffices to find conditions under which $\ker \mathscr{S}_\nabla^{gr} = 0$.

Suppose $\xi_i: E_{\xi_i} \to M$, $i = 1, 2, 3$, are linear bundles satisfying

$$\mathscr{F}(\mathscr{Y}, \xi_1) = [\kappa], \qquad \mathscr{F}(\mathscr{Y}, \xi_2) = [P], \mathscr{F}(\mathscr{Y}, \xi_3) = \bar{\Lambda}^q(\mathscr{Y}).$$

Then

$$h(\mathscr{S}_\nabla^{gr}(\mathscr{C}\,\mathrm{smbl}_i\,\square) = h(\mathscr{C}\,\mathrm{smbl}_i\,\square) \circ h_\nabla,\ h_\nabla = h(\mathscr{C}\,\mathrm{smb}_k\,\nabla).$$

Therefore $\ker \mathscr{S}_\nabla^{gr} = 0$ if the set

$$\mathrm{char}\,\nabla^* = \{\theta \in T^*(M, \mathscr{Y}_\infty) \mid \mathrm{coker}\, h_{\nabla,\theta} \neq 0\}$$

is nowhere dense in $T^*(M, \mathscr{Y}_\infty)$. Thus we have the following:

PROPOSITION. *If the equation $\mathscr{Y}$ is regular with respect to the operator ∇ and the set* $\mathrm{char}\,\nabla^*$ *is nowhere dense, we have* $\ker v_\nabla^{(1,q)} = 0$.

Proof. The previous arguments prove the statement within an arbitrary affine chart, from which it follows globally. ▮

If the dimension of the projective module $[P]$ is no greater than that of the projective module $[\kappa]$ (this is equivalent to the equation being non-overdetermined) the assumptions of Proposition 10.8 hold in the "general position" situation. Indeed, if $m = \dim[\kappa]$ and $l = \dim[P]$, then in local coordinates the operator ∇ is expressed by the $l \times m$-matrix $\|\nabla_{ij}\|$, $\nabla_{ij} \in \mathscr{C}\,\mathrm{Diff}\,\mathscr{F}(\mathscr{Y})$. Therefore the homomorphism h_∇ is given by the matrix $s(\nabla) = \|\mathscr{C}\,\mathrm{smbl}_k\,\nabla_{ij}\|$ and, if $l \leqslant m$, the condition $\theta \in \mathrm{cher}\,\nabla^*$ means that all the minors of order l of the matrix $s(\nabla)$ vanish at the point θ. Thus, in this situation we have generally $\mathrm{codim}\,\mathrm{char}\,\nabla^* \geqslant 1$.

The above gives us a simple method for finding the set $\mathrm{char}\,\nabla^*$. To do this we write out the elements of the matrix $s(\nabla)$ in the coordinates introduce in 7.4 and express the fact that its rank must be $\leqslant l$.

10.9. Beginning at this point, we will consider certain methods for estimating or computing the cohomology of the complexes $K^{p,*}$, $p > 1$. In particular, in the process we shall indicate convenient conditions under which the two line theorem is valid. These methods may also be effective in the study of concrete equations. In the present subsection we will consider the preparatory question of finding the cohomology of certain complexes

described below. All the arguments will be carried out for a certain object of the category DE and $\mathscr{F}$ we will denote its algebra of functions.

Suppose Q_i, $i = 1,..., k$, are projective $\mathscr{F}$-modules. Identifying the $\mathscr{F}$-module $\mathscr{C}\ \mathrm{Diff}(Q_1,..., Q_k; \bar{\Lambda}^q)$ with the tensor product (over $\mathscr{F}$)

$$\mathscr{C}\ \mathrm{Diff}(Q_1, \mathscr{F}) \otimes \cdots \otimes \mathscr{C}\ \mathrm{Diff}(Q_k, \mathscr{F}) \otimes \bar{\Lambda}^q$$

and choosing a number i, $1 \leqslant i \leqslant k$, let us introduce into the complex $\mathscr{C}\ \mathrm{Diff}\{Q_1,..., Q_k\}$ (see 9.5.) an increasing filtration for each integer $1 \leqslant i \leqslant k$ by saying that cochains of filtration $\leqslant p$ are tensor products of the form

$$\begin{aligned} A^{p,q} = \mathscr{C}\ \mathrm{Diff}(Q_1, \mathscr{F}) \otimes \cdots \otimes \mathscr{C}\ \mathrm{Diff}(Q_{i-1}, \mathscr{F}) \otimes \mathscr{C}\ \mathrm{Diff}_{p+q}(Q_i, \mathscr{F}) \\ \otimes \mathscr{C}\ \mathrm{Diff}(Q_{i+1}, \mathscr{F}) \otimes \cdots \otimes \mathscr{C}\ \mathrm{Diff}(Q_k, \mathscr{F}) \otimes \bar{\Lambda}^q. \end{aligned}$$

In the case when $i = k = 1$, this filtration coincides with the filtration of the complex $\widetilde{\mathscr{S}Q}$ $(Q = Q_1)$ by the subcomplexes $\widetilde{\mathscr{S}Q_p}$ (see 4.2). Consider the spectral sequence generated by this filtration. Then the components $B_0^{p,q}$ of filtration p and degree q of the zeroth term of this spectral sequence are of the form

$$\begin{aligned} B_0^{p,q} &= A^{p,q}/A^{p-1,q} \\ &= \mathscr{C}\ \mathrm{Diff}(Q_1,..., Q_{i-1}, Q_{i+1},..., Q_k; \mathscr{F}) \otimes \mathscr{C}\ \mathrm{smbl}_{p+q}(Q_i, \bar{\Lambda}^q) \end{aligned}$$

while the zeroth differential, mapping $B_0^{p,q}$ into $B_0^{p,q+1}$, has the form $\mathrm{id} \otimes \delta_{p,q}$, where

$$\delta_{p,q}: \mathscr{C}\ \mathrm{smbl}_{p+q}(Q_i, \bar{\Lambda}^q) \to \mathscr{C}\ \mathrm{smbl}_{p+q+1}(Q_i, \bar{\Lambda}^{q+1})$$

is a similar differential in the spectral sequence of the complex $\widetilde{\mathscr{S}Q_i} = \mathscr{C}\ \mathrm{Diff}_{(1)}\{Q_i\}$ with respect to the filtration described above. The operators $\delta = \delta_{p,q}$ are one of the variants of the Spencer δ-operators (see [33, 36]) and are obviously $\mathscr{F}$-module homomorphisms. Then the complexes

$$\begin{aligned} 0 \longrightarrow \mathscr{C}\ \mathrm{smbl}_p(Q, \mathscr{F}) \xrightarrow{\delta_{p,0}} \mathscr{C}\ \mathrm{smbl}_{p+1}(Q, \bar{\Lambda}^1) \xrightarrow{\delta_{p,1}} \cdots \\ \xrightarrow{\delta_{p,n-1}} \mathscr{C}\ \mathrm{smbl}_{p+n}(Q, \bar{\Lambda}^n) \to 0 \end{aligned}$$

are acyclic for $p > -n$ (we agree that $\mathscr{C}\ \mathrm{smbl}_p\, Q = 0$ if $p < 0$) and therefore the first term of the spectral sequence described above consists of components, except the component of zeroth filtration and degree n, which equals

$$\begin{aligned} &\mathscr{C}\ \mathrm{Diff}(Q_1,..., Q_{i-1}, Q_{i+1},..., Q_k; \mathscr{F}) \otimes \mathscr{C}\ \mathrm{smbl}_0(Q_i, \bar{\Lambda}^n) \\ &\quad = \mathscr{C}\ \mathrm{Diff}(Q_1,..., Q_{i-1}, Q_{i+1},..., Q_k; \mathscr{F}) \otimes \hat{Q}_i \\ &\quad = \mathscr{C}\ \mathrm{Diff}(Q_1,..., Q_{i-1}, Q_{i+1},..., Q_k; \hat{Q}_i) = H^n(\mathscr{C}\ \mathrm{Diff}\{Q_1,..., Q_k\}). \end{aligned}$$

For the sequel we will need subcomplexes of the complex $\mathscr{C}\,\mathrm{Diff}\{Q_1,\dots,Q_k\}$ of the following form. Suppose

$$\mathscr{O}' \subset \mathscr{C}\,\mathrm{Diff}(Q_1,\mathscr{F}) \otimes \cdots \otimes \mathscr{C}\,\mathrm{Diff}(Q_{i-1},\mathscr{F}),\ \mathscr{O} \subset \mathscr{C}\,\mathrm{Diff}(Q_i,\mathscr{F})$$
$$\mathscr{O}'' \subset \mathscr{C}\,\mathrm{Diff}(Q_{i+1},\mathscr{F}) \otimes \cdots \otimes \mathscr{C}\,\mathrm{Diff}(Q_k,\mathscr{F})$$

are submodules such that $\mathscr{O}' \otimes \bar{\Lambda}^*$, $\mathscr{O}'' \otimes \bar{\Lambda}^*$, and $\mathscr{O} \otimes \bar{\Lambda}^*$ are subcomplexes in $\mathscr{C}\,\mathrm{Diff}\{Q_1,\dots,Q_{i-1}\}$, $\mathscr{C}\,\mathrm{Diff}\{Q_{i+1},\dots,Q_k\}$ and $\mathscr{C}\,\mathrm{Diff}_{(1)}\{Q_i\}$, respectively. Then $\mathscr{O}' \otimes \mathscr{O} \otimes \mathscr{O}'' \otimes \bar{\Lambda}^*$ is a subcomplex in $\mathscr{C}\,\mathrm{Diff}\{Q_1,\dots,Q_k\}$ which shall be denoted by $\{\mathscr{O}',\mathscr{O},\mathscr{O}''\}$. If the submodules $\mathscr{O}_j \subset \mathscr{C}\,\mathrm{Diff}(Q_j,\mathscr{F})$ satisfy $\mathscr{O}' = \mathscr{O}_1 \otimes \cdots \otimes \mathscr{O}_{i-1}$, $\mathscr{O} = \mathscr{O}_i$ and $\mathscr{O}'' = \mathscr{O}_{i+1} \otimes \cdots \otimes \mathscr{O}_k$, then the complex $\{\mathscr{O}',\mathscr{O},\mathscr{O}''\}$ will be denoted by $\{\mathscr{O}_1,\dots,\mathscr{O}_k\}$.

The ith filtration in the complex $\mathscr{C}\,\mathrm{Diff}\{Q_1,\dots,Q_k\}$ induces a filtration (also "ith") in the complex $\{\mathscr{O}',\mathscr{O},\mathscr{O}''\}$ (resp. $\{\mathscr{O}_1,\dots,\mathscr{O}_k\}$) and we can consider the spectral sequence corresponding to this filtration. If the modules $\mathscr{O}'$ and $\mathscr{O}''$ (resp. $\mathscr{O}_s$, $s \neq i$) are projective, then, as above, the components $B_0^{p,q} = B_0^{p,q}(\mathscr{O}',\mathscr{O},\mathscr{O}'')$ (resp. $B_0^{p,q} = B_0^{p,q}(\mathscr{O}_1,\dots,\mathscr{O}_k)$) of filtration p and degree q of the zeroth term of this spectral sequence are of the form

$$B_0^{p,q}(\mathscr{O}',\mathscr{O},\mathscr{O}'') = \mathscr{O}' \otimes \mathscr{O}'' \otimes sm_{p+q}(\mathscr{O},\bar{\Lambda}^q),$$
$$\begin{aligned} B_0^{p,q}(\mathscr{O}_1,\dots,\mathscr{O}_k) &= \mathscr{O}_1 \otimes \cdots \otimes \mathscr{O}_{i-1} \otimes \mathscr{O}_{i+1} \otimes \cdots \otimes \mathscr{O}_k \otimes sm_{p+q}(\mathscr{O}_i,\bar{\Lambda}^q) \\ &= \mathscr{O}^{(i)} \otimes sm_{p+q}(\mathscr{O}_i,\bar{\Lambda}^q), \end{aligned}$$

where

$$\mathscr{O}^{(i)} = \mathscr{O}_1 \otimes \cdots \otimes \mathscr{O}_{i-1} \otimes \mathscr{O}_{i+1} \otimes \cdots \otimes \mathscr{O}_k,$$
$$\begin{aligned} &sm_{p,q}(\mathscr{O},\bar{\Lambda}^q) \\ &\quad = [\mathscr{O} \otimes \bar{\Lambda}^q \cap \mathscr{C}\,\mathrm{Diff}_{p+q}(Q_i,\bar{\Lambda}^q)]/[\mathscr{O} \otimes \bar{\Lambda}^q \cap \mathscr{C}\,\mathrm{Diff}_{p+q-1}(Q_i,\bar{\Lambda}^q)]. \end{aligned}$$

Note that $sm_j(\mathscr{O},\bar{\Lambda}^q) \subset \mathscr{C}\,\mathrm{smbl}_j(Q_i,\bar{\Lambda}^q)$ and $\delta_{p,q}(sm_{p+q}(\mathscr{O},\bar{\Lambda}^q) \subset sm_{p+q+1}(\mathscr{O},\bar{\Lambda}^q)$. The differential in the zeroth term of the spectral sequence under consideration maps $B_0^{p,q}$ into $B_0^{p,q+1}$ and is of the form $\mathrm{id}_{\mathscr{O}' \otimes \mathscr{O}''} \otimes \delta_{p,q}$(resp. $\mathrm{id}_{\mathscr{O}^{(i)}} \otimes \delta_{p,q}$). For this reason the component $B_1^{p,q} = B_1^{p,q}(\mathscr{O}',\mathscr{O},\mathscr{O}'')$ (resp. $B_1^{p,q} = B_1^{p,q}(\mathscr{O}_1,\dots,\mathscr{O}_k)$) of the first term of the spectral sequence with filtration p and degree q is of the form

$$B_1^{p,q}(\mathscr{O}',\mathscr{O},\mathscr{O}'') = \mathscr{O}' \otimes \mathscr{O}'' \otimes h^{p,q}(\mathscr{O}), \tag{10.9.1}$$

respectively,

$$\begin{aligned} B_1^{p,q}(\mathscr{O}_1,\dots,\mathscr{O}_k) &= \mathscr{O}_1 \otimes \cdots \otimes \mathscr{O}_{i-1} \otimes \mathscr{O}_{i+1} \otimes \cdots \otimes \mathscr{O}_k \otimes h^{p,q}(\mathscr{O}_i) \\ &= \mathscr{O}^{(i)} \otimes h^{p,q}(\mathscr{O}_i), \end{aligned}$$

where $h^{p,q}(\mathcal{O})$ is the module of q-dimensional homology of the complex $sm_p(\mathcal{O})$,

$$0 \longrightarrow sm_p(\mathcal{O}, \mathcal{F}) \xrightarrow{\delta_{p,q}} \cdots \xrightarrow{\delta_{p,q-1}} sm_{p+q}(\mathcal{O}, \bar{\Lambda}^q) \xrightarrow{\delta_{p,q}} \cdots$$
$$\xrightarrow{\delta_{p,n-1}} sm_{p+n}(\mathcal{O}, \bar{\Lambda}^n) \to 0.$$

Now let us note the following immediate consequence of the spectral sequence discussed above.

PROPOSITION. *Suppose both the module, $\mathcal{O}'$, $\mathcal{O}''$ are projective, while the complexes $sm_p(\mathcal{O})$ are acylcic in dimensions $<s$. Then the complex $\{\mathcal{O}', \mathcal{O}, \mathcal{O}''\}$ is acyclic in the same dimension.*

Proof. The assumptions of the proposition mean $h^{p,q}(\mathcal{O}) = 0$ if $q < s$. Hence it follows from (10.9.1), $B_1^{p,q}(\mathcal{O}', \mathcal{O}, \mathcal{O}'') = 0$ if $q < s$. In its turn, this implies that the components of filtration p and degree q of the limit term of the spectral sequence are also trivial if $q < s$. ▮

COROLLARY. *If the modules $\mathcal{O}'$ and $\mathcal{O}''$ are projective and there exists a number l_0 such that $h^{l,q}(\mathcal{O}) = 0$ whenever $(l, q) \neq (l_0, n)$, then the complex $\{\mathcal{O}', \mathcal{O}, \mathcal{O}''\}$ is acyclic in dimensions differing from n and*

$$H^n(\{\mathcal{O}', \mathcal{O}, \mathcal{O}''\}) = \mathcal{O}' \otimes \mathcal{O}'' \otimes h^{l_0,n}(\mathcal{O}).$$

Remark. The assumptions of this corollary obviously hold if $\mathcal{O} = \mathscr{C}\,\mathrm{Diff}(Q_i, \mathcal{F})$.

10.10. In the sequel, the role of the submodules $\mathcal{O}_i \subset \mathscr{C}\,\mathrm{Diff}(Q_i, \mathcal{F})$ will be played by the images of maps of the form

$$\mathscr{C}\,\mathrm{Diff}(Q_i', \mathcal{F}) \to \mathscr{C}\,\mathrm{Diff}(Q_i, \mathcal{F}), \qquad \square \mapsto \square \circ \nabla,$$

where $\nabla \in \mathscr{C}\,\mathrm{Diff}(Q_i, Q_i')$ is fixed, $\square \in \mathscr{C}\,\mathrm{Diff}(Q', \mathcal{F})$. In this case the subcomplex $\mathcal{O} \otimes \bar{\Lambda}^* \subset \widetilde{\mathscr{S}Q_i}$ may be supplied with a filtration, which we shall call the left filtration which differs from the one introduced in the previous subsection; the latter shall now be called the right filtration. To do this, note that $\mathcal{O} \otimes \bar{\Lambda}^*$ is the image of the complex $\widetilde{\mathscr{S}Q_i'}$ by the map $\mathscr{S}_\nabla$ (see 4.2.) Hence the filtration of the complex $\widetilde{\mathscr{S}Q_i'}$ by the complexes $\widetilde{\mathscr{S}_k Q_i'}$ is carried over by the map $\mathscr{S}_\nabla$ into $\mathcal{O} \otimes \bar{\Lambda}^*$. Obviously, elements of degree q and filtration $\leqslant p$ constitute a submodule in $\mathcal{O} \otimes \bar{\Lambda}^*$ of the form

$$\mathcal{O}_{(p+q)} \otimes \bar{\Lambda}^q, \ \mathcal{O}_{(p+q)} = \{\square \circ \nabla \mid \square \in \mathscr{C}\,\mathrm{Diff}_{p+q}(Q_i', \mathcal{F})\}.$$

If $\operatorname{ord} \nabla = k$, then $\mathcal{O}_{(p+q)} \otimes \bar{\Lambda}^q \subset \{\mathscr{C}\,\mathrm{Diff}_{p+q+k}(Q, \mathcal{F}) \cap \mathcal{O}\} \otimes \bar{\Lambda}^q$ and this

last module consists of elements of degree q and of right filtration $\leqslant p+k$, i.e., the left filtration is included in the right one, the filtration index being increased by k.

Consider the spectral sequence generated by the left filtration in $\mathscr{O} \otimes \bar{\Lambda}^*$. Then the components of its zeroth term consisting of elements of degree q and filtration p, are of the form

$$D_0^{p,q} = D_0^{p,q}(\mathscr{O}) = [\mathscr{O}_{(p+q)}/\mathscr{O}_{(p+q-1)}] \otimes \bar{\Lambda}^q.$$

The zeroth differential of this spectral sequence, further denoted by $\partial = \partial_{p,q}$, maps $D_0^{p,q}$ into $D_0^{p,q+1}$, so that its zeroth term splits into the complexes $\mathrm{sml}_p(\mathscr{O})$:

$$0 \longrightarrow D_0^{p,0} \xrightarrow{\partial_{p,0}} \cdots \xrightarrow{\partial_{p,q-1}} D_0^{p,q} \xrightarrow{\partial_{p,q}} \cdots \xrightarrow{\partial_{p,n-1}} D_0^{p,n} \to 0.$$

Obviously $\partial_{p,q}$ is a $\mathscr{F}$-module homomorphism.

The cohomology of the complex $\mathrm{sml}_p(\mathscr{O})$ in the term $D_0^{p,q}$ is denoted by $H^{p,q}(\mathscr{O})$. Thus $H^{p,q}(\mathscr{O}) = D_1^{p,q}(\mathscr{O})$, where $D_1^{p,q}(\mathscr{O})$ is the component of degree q and filtration p of the first term of the spectral sequence under consideration.

The left filtration in the complex $\mathscr{O} \otimes \bar{\Lambda}^*$ induces a filtration, called the left filtration of ith number, as well in the complex $\{\mathscr{O}', \mathscr{O}, \mathscr{O}''\}$ (resp. $\{\mathscr{O}_1, \ldots, \mathscr{O}_k\}$) with respect to which elements of degree q and filtration $\leqslant p$ constitute the submodule $\mathscr{O}' \otimes \mathscr{O}_{(p+q)} \otimes \mathscr{O}'' \otimes \bar{\Lambda}^q$, respectively, $\mathscr{O}_1 \otimes \cdots \otimes \mathscr{O}_{i-1} \otimes \mathscr{O}_{(p+q)} \otimes \mathscr{O}_{i+1} \otimes \cdots \otimes \mathscr{O}_k \otimes \bar{\Lambda}^q$.

Consider the spectral sequence by this filtration. The components of its zeroth term of degree q and filtration p in the case when the modules $\mathscr{O}'$, $\mathscr{O}$ (resp. $\mathscr{O}_s$, $i \neq s$) are projective, will be of the form

$$\begin{aligned} D_0^{p,q}(\mathscr{O}', \mathscr{O}, \mathscr{O}'') &= \mathscr{O}' \otimes \mathscr{O}'' \otimes D_0^{p,q}(\mathscr{O}), \\ D_0^{p,q}(\mathscr{O}_1, \ldots, \mathscr{O}_k) &= \mathscr{O}_1 \otimes \cdots \otimes \mathscr{O}_{i-1} \otimes \mathscr{O}_{i+1} \otimes \cdots \otimes \mathscr{O}_k \otimes D_0^{p,q}(\mathscr{O}) \\ &= \mathscr{O}^{(i)} \otimes D_0^{p,q}(\mathscr{O}). \end{aligned}$$

As before, the differentials of the zeroth term of this spectral sequence have the form $\mathrm{id}_{\mathscr{O}' \otimes \mathscr{O}''} \otimes \partial_{p,q}$ (resp. $\mathrm{id}_{\mathscr{O}^{(i)}} \otimes \partial_{p,q}$) so that the components of its first term of degree q and filtration p are the form

$$D_1^{p,q}(\mathscr{O}', \mathscr{O}, \mathscr{O}'') = \mathscr{O}' \otimes \mathscr{O}'' \otimes H^{p,q}(\mathscr{O}),$$

respectively,

$$D_1^{p,q}(\mathscr{O}_1, \ldots, \mathscr{O}_k) = \mathscr{O}^{(i)} \otimes H^{p,q}(\mathscr{O}).$$

As in previous subsection, we have

PROPOSITION. *Suppose the modules* $\mathcal{O}'$, $\mathcal{O}''$ *are projective while the complexes* $\mathrm{sml}_p(\mathcal{O})$ *are acyclic in dimensions* $<s$. *Then the complex* $\{\mathcal{O}', \mathcal{O}, \mathcal{O}''\}$ *is also acyclic in dimensions* $<s$.

COROLLARY. *If the modules* $\mathcal{O}'$ *and* $\mathcal{O}''$ *are projective and there exists a number* l_0 *such that* $H^{l,q}(\mathcal{O}) = 0$ *whenever* $(l, q) \neq (l_0, n)$, *then the complex* $\{\mathcal{O}', \mathcal{O}, \mathcal{O}''\}$ *is acyclic in dimensions differing from n and*

$$H^n(\{\mathcal{O}', \mathcal{O}, \mathcal{O}''\}) = \mathcal{O}' \otimes \mathcal{O}'' \otimes H^{l_0,n}(\mathcal{O}).$$

Remark. The assumptions of this corollary obviously hold if $\ker \mathcal{S}_\nabla = 0$. In this case $l_0 = -n$.

As we shall see in the sequel, the left (resp. right) filtration is convenient in the case when the kernel (resp. the image) of the map $\mathcal{S}_\nabla$ is "simple.'

10.11. Returning to the situation with which we are concerned, put $R = K_0^{1,0}$ and recall that the module $K_0^{1,0}$ is the image of the map $\mathscr{C}\,\mathrm{Diff}(\nabla, id_{\mathcal{F}})$ (see 10.5). Let us also introduce the following notation.

Suppose Π is the set of finite sequences consisting of 0 and 1. The "empty" sequence, which has zero length, is also viewed as an element of the set Π. We shall say that the sequence $\sigma \in \Pi$ is trivial if it is empty or if it consists of zeros only. If $\sigma, \tau \in \Pi$, $\sigma = (\varepsilon_1, \ldots, \varepsilon_k)$, $\tau = (\varepsilon'_1, \ldots, \varepsilon'_l)$, we put $|\sigma| = k$, $\sigma + \tau = (\varepsilon_1, \ldots \varepsilon_k, \varepsilon'_1, \ldots \varepsilon'_l)$ and if $|\sigma| = |\tau|$, we put $\sigma\tau = (\varepsilon_1 \cdot \varepsilon'_1, \ldots, \varepsilon_k \cdot \varepsilon'_k)$. To any sequence $\sigma = (\varepsilon_1, \ldots, \varepsilon_k) \in \Pi$ we assign the subcomplex $C(\sigma) = C(\sigma, R)$ of the complex $\mathscr{C}\,\mathrm{Diff}_{(k)}\{[\kappa]\}$ by putting $C(\sigma) = \{\mathcal{O}_1, \ldots, \mathcal{O}_k\}$, where $\mathcal{O}_i = \mathscr{C}\,\mathrm{Diff}([\kappa], \mathcal{F})$ if $\varepsilon_i = 1$ and $\mathcal{O}_i = R$ if $i = 0$. If the submodule $R \subset \mathscr{C}\,\mathrm{Diff}([\kappa], \mathcal{F})$ possesses a direct complement and $|\sigma| = |\tau|$, then obviously $C(\sigma) \cap C(\tau) = C(\sigma\tau)$.

Suppose the sequence $\sigma_{i,p} = (\varepsilon_1, \ldots, \varepsilon_p) \in \Pi$ satisfies $\varepsilon_s = 1$ if $s \neq i$ and $\varepsilon_i = 0$. Consider the subcomplex $C_p(R) = \sum_{i=1}^{p} C(\sigma_{i,p})$ of the complex $\mathscr{C}\,\mathrm{Diff}_{(p)}\{[\kappa]\}$. It is obviously invariant with respect to the action of the permutation group S_p in $\mathscr{C}\,\mathrm{Diff}_{(p)}\{[\kappa]\}$ and its image under the alternation operation coincides with $K_0^{p,*}$. For this reason, the complex $K_0^{p,*}$ is a direct summand in $C_p(R)$ and

$$H^s(K_0^{p,*}) = H^s_{\mathrm{alt}}(C_p(R)), \tag{10.11.1}$$

where $H^s_{\mathrm{alt}}(C_p(R))$ is the anti-symmetric part of the group $H^s(C_p(R))$ with respect to the induced action of the group S_p in $\mathscr{C}\,\mathrm{Diff}_{(p)}\{[\kappa]\}$. In particular, the fact that the complex $K_0^{p,*}$ is acyclic in some dimension follows from a similar property of the complex $C_p(R)$. For this reason, we shall study the cohomology of the latter. Further, by a complex of type s, $s \leqslant n$, we mean a complex which is acyclic in dimensions less than s and greater than n.

PROPOSITION. (1) *Suppose* $h^{l,q}(K_0^{1,0}) = 0$, *respectively*, $H^{l,q}(K_0^{1,0}) = 0$, *when* $q \leqslant s \leqslant n$ *and* $K_0^{1,0}$ *possesses a direct complement in* $\mathscr{C}\,\mathrm{Diff}([\kappa], \mathscr{F}(\mathscr{Y}))$. *Then the complexes* $C_p(K_0^{1,0})$, $p > 1$, *are of the* sth *type.*

(2) *Suppose* $K_0^{1,0}$ *possesses a complement in* $\mathscr{C}\,\mathrm{Diff}([\kappa], \mathscr{F}(\mathscr{Y}))$, *there exists a number* l_0 *such that* $h^{l,q}(K_0^{1,0}) = 0$ (*resp.* $H^{l,q}(K_0^{1,0}) = 0$) *if* $(l, q) \neq (l_0, n)$ *and the natural inclusion* $C((0, \varepsilon)) \hookrightarrow C((1, \varepsilon))$, $\varepsilon = 0, 1$, *induces the monomorphism* $H^n(C((0, \varepsilon))) \to H^n(C((1, \varepsilon)))$. *Then the complexes* $C_p(K_0^{1,0})$ *are of type* n.

Proof. We shall carry out the proof using the assumption which concerns $h^{p,q}$. Replacing in the appropriate places references to Proposition 10.9 (resp. Corollary 10.9) by references to Proposition 10.10 (resp. to Corollary 10.10 we obtain the proof in the case when the assumptions concern $H^{p,q}$.

Consider the complexes

$$C_{p,\tau} = C_{p,\tau}(R) = \sum_{i=1}^{p} C(\sigma_{i,p} + \tau, R) \subset \mathscr{C}\,\mathrm{Diff}_{(p+|\tau|)}\{[\kappa]\}.$$

Then $C_{p,\tau}(R) = C_p(R)$ if $\tau = \varnothing$. Let us agree to denote by $[l]$ the sequence of length l constituted by units only and, for $\varepsilon = 0$ or 1 and $\tau \in \Pi$, to write $\varepsilon + \tau$ instead of $(\varepsilon) + \tau$. In these notations, we have

$$C_{p+1,\tau} = C_{p,1+\tau} + C([p] + 0 + \tau), \qquad C_{p,1+\tau} \cap C([p] + 0 + \tau) = C_{p,0+\tau}. \tag{10.11.2}$$

All the complexes which appear in this relation are viewed as subcomplexes of $\mathscr{C}\,\mathrm{Diff}_{(p+1+|\tau|)}\{[\kappa]\}$. The second relation is valid because of the fact that $R = K_0^{1,0}$ possesses a direct complement in $\mathscr{C}\,\mathrm{Diff}([\kappa], \mathscr{F})$ and therefore $C(\sigma) \cap C(\sigma') = C(\sigma\sigma')$.

The complex $C_{p,\varepsilon+\tau}$, $\varepsilon = 0$ or 1, is of the form $\{\mathscr{O}', \mathscr{O}, \mathscr{O}''\}$,

$$\mathscr{O}' = \sum_{i=1}^{p} A_1 \otimes \cdots \otimes A_{i-1} \otimes R \otimes A_{i+1} \otimes \cdots \otimes A_p,$$

$$A_j = \mathscr{C}\,\mathrm{Diff}([\kappa], \mathscr{F}(\mathscr{Y})),$$

$$\mathscr{O}'' = B_1 \otimes \cdots \otimes B_{|\tau|}, \qquad \tau = (\varepsilon_1, \ldots, \varepsilon_{|\tau|}),$$

where $B_j = \mathscr{C}\,\mathrm{Diff}([\kappa], \mathscr{F}(\mathscr{Y}))$ if $\varepsilon_j = 1$ and $B_j = R$ if $\varepsilon_j = 0$. Here $\mathscr{O}$ equals $\mathscr{C}\,\mathrm{Diff}([\kappa], \mathscr{F}(\mathscr{Y}))$ (resp. R) if $\varepsilon = 1$ (resp. $\varepsilon = 0$). For this reason Proposition 10.9 and its corollary imply that the complex $C_{p,0+\tau}$ is of the

$(s+1)$th type if assumption 1)) holds while the complex $C_{p,1+\tau}$ is of type n. The complex $C([p]+0+\tau)$ is also of type n. Now note that

$$\begin{aligned} C_{p+1,\tau}/C_{p,1+\tau} &= C([p]+0+\tau)/C_{p,1+\tau} \cap C([p]+0+\tau) \\ &= C([p]+0+\tau)/C_{p,0+\tau} \stackrel{\text{def}}{=} C'_{p,\tau}. \end{aligned} \tag{10.11.3}$$

If assumption (1) holds, the exact cohomology sequence of the pair $C_{p,0+\tau} \subset C([p]+0+\tau)$ implies that the quotient complex $C'_{p,\tau}$ is of type s, while the sequence for the pair $C_{p,1+\tau} \subset C_{p+1,\tau}$, by (10.11.3), implies in its turn that $C_{p+1,\tau}$ is of type s. This proves statement (1).

Statement (2) will be proved by induction over $p \geqslant 1$.

Having in mind (10.11.3) let us consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 \to C_{p,1+\tau} \to & C_{p+1,\tau} & \to \dfrac{C_{p+1,\tau}}{C_{p,1+\tau}} = \dfrac{C([p]+0+\tau)}{C_{p,0+\tau}} \to 0 \\ \downarrow{=} & \downarrow{\alpha} & \downarrow{\beta} \\ 0 \to C_{p,1+\tau} \to & C([p+1]+\tau) & \to \dfrac{C([p+1]+\tau)}{C_{p,1+\tau}} \to 0, \end{array} \tag{10.11.4}$$

where α is the natural inclusion. The sequence

$$0 \to \frac{C([p]+0+\tau)}{C_{p,0+\tau}} \to \frac{C([p+1]+\tau)}{C_{p,1+\tau}} \to \frac{C([p+1]+\tau)}{C([p]+0+\tau)} \to 0 \tag{10.11.5}$$

is also exact.

We shall show by induction that complexes $C_{p,\sigma}$ are of the type n and homomorphism $H^n(C_{p,\sigma}) \to H^n(C([p]+\sigma))$ ($|\sigma|>0$, if $p=1$) induced by the inclusion $C_{p,\sigma} \hookrightarrow C([p]+\sigma)$ is injective.

First, note that $C_{1,\sigma} = C(0+\sigma)$ and complexes $C(\tau)$, $|\tau|>0$, are of the type n. Further, injectivity of the homomorphism $H^n(C((0,\varepsilon))) \to H^n(C((1,\varepsilon)))$, $\varepsilon = 0, 1$, implies the injectivity of homomorphisms $H^n(C(0+\varepsilon+\tau)) \to H^n(C(1+\varepsilon+\tau))$. This follows evidently from Corollary 10.9. Hence, it is possible to begin the induction.

Assuming the induction hypothesis for p let us consider the exact cohomology sequence of the pair $C_{p,0+\tau} \subset C([p]+0+\tau)$ which implies that the complex $C([p]+0+\tau)/C_{p,0+\tau}$ has the type n. In its turn this fact and the exact cohomology sequence, corresponding to the upper row of the diagram (10.11.4) show that the complex $C_{p+1,\tau}$ has the type n. Similarly, one can find, using exact cohomology sequence of the pair $C_{p,1+\tau} \subset$

$C([p+1]+\tau) = C([p]+1+\tau)$ that the complex $C([p+1]+\tau)/C_{p,1+\tau}$ is of the type n.

Above considerations and exact cohomology sequences, corresponding to rows of the diagram (10.11.4) now lead to the commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 \to H^n(C_{p,1+\tau}) \to & & H^n(C_{p+1,\tau}) & \to H^n\left(\dfrac{C([p]+0+\tau)}{C_{p,0+\tau}}\right) \to 0 \\
\Big\downarrow = & & \Big\downarrow H^n(\alpha) & \Big\downarrow H^n(\beta) \\
0 \to H^n(C_{p,1+\tau}) \to & & H^n(C([p+1]+\tau)) & \to H^n\left(\dfrac{C([p+1]+\tau)}{C_{p,1+\tau}}\right) \to 0.
\end{array}$$

Evidently, the injectivity of $H^n(\alpha)$ is implied by the injectivity of $H^n(\beta)$. The latter as one can see from the exact cohomology sequence, corresponding to (10.11.5) is a consequence of the fact that $C([p+1]+\tau)/C_{([p]+0+\tau)}$ has the type n. But this follows from the exact cohomology sequence of the pair $C([p]+0+\tau) \subset C([p+1]+\tau)$ and the induction hypothesis for $p=1$. ∎

Comparing the proposition that we have proved with (10.11.1) and (10.7.2), we obtain the following result which estimates the cohomology of the complexes $K^{p,*}$ and therefore, the term $E_1(\mathscr{Y})$.

THEOREM. (1) *Suppose* $h^{l,q}(K_0^{1,0}) = 0$ (*resp.* $H^{l,q}(K_0^{1,0}) = 0$) *when* $q \leqslant s$ ($s \leqslant n$) *and* $K_0^{1,0}$ *has a direct complement in* $\mathscr{C}$ $\mathrm{Diff}([\kappa], \mathscr{F}(\mathscr{Y}))$. *Then* $E_1^{1,q}(\mathscr{Y}) = H^q(K^{1,*}) = 0$ *if* $q < s$ *while* $E_1^{p,q} = H^q(K^{p,*}) = 0$ *if* $q < s-1$, $p > 1$.

(2) *Suppose* $K_0^{1,0}$ *possesses a direct complement in* $\mathscr{C}$ $\mathrm{Diff}([\kappa], \mathscr{F}(\mathscr{Y}))$, *there exists a number* l_0 *such that* $h^{p,q}(K_0^{1,0}) = 0$ (*resp.* $H^{p,q}(K_0^{1,0}) = 0$) *if* $(l, q) \neq (l_0, n)$ *and suppose the natural inclusion* $C((0,\varepsilon)) \hookrightarrow C((1,\varepsilon))$ *induces a monomorphism* $H^n(C((0,\varepsilon)) \to H^n(C((1,\varepsilon)))$, $\varepsilon = 0, 1$. *Then for* $p \geqslant 1$, *we have* $E_1^{p,q}(\mathscr{Y}) = H^q(K^{p,*}) = 0$ *if* $q < n-1$ *or* $q > n$.

10.12. Now let us consider a very important particular case which almost always takes place for non-overdetermined equation. Namely, suppose that $\ker \mathscr{S}_\nabla = \ker v_\nabla^{(1)} = 0$ (see 10.8). In this case, the complex $K_0^{1,*}$ is isomorphic to the complex $\widetilde{\mathscr{S}[P]}$ and its left filtration is the image of the standard filtration $\{\widetilde{\mathscr{S}_k[P]}\}$ in $\widetilde{\mathscr{S}[P]}$. Hence, for the number l_0 which appears in Theorem 10.11 we can take $-n$ (see Remark 10.10).

Further, since $\ker \mathscr{S}_\nabla = 0$, we have

$$C((0,\varepsilon)) = \mathscr{C}\,\mathrm{Diff}\{[P], [Q]\} = \mathscr{C}\,\mathrm{Diff}([P], \widetilde{\mathscr{S}[Q]}), \qquad Q = P \text{ or } \kappa,$$

$$C((1,\varepsilon)) = \mathscr{C}\,\mathrm{Diff}\{[\kappa], [Q]\} = \mathscr{C}\,\mathrm{Diff}([\kappa], \widetilde{\mathscr{S}[Q]}), \qquad Q = P \text{ or } \kappa,$$

and the natural inclusion $C((0,\varepsilon)) \to C((1,\varepsilon))$ can be identified with the map $\square \mapsto \square \circ \nabla$, where $\square \in \mathscr{C}\,\mathrm{Diff}([P], \widetilde{\mathscr{S}[Q]})$. In view of this, the corresponding map $H^n(C((0,\varepsilon))) \to H^n(C((1,\varepsilon)))$ is identified with the map $\mathscr{C}\,\mathrm{Diff}(\nabla, \mathrm{id}_{[\hat{P}]})$ (see 9.5):

$$\begin{aligned} &H^n(\mathscr{C}\,\mathrm{Diff}\{[P],[Q]\}) \\ &\quad = \mathscr{C}\,\mathrm{Diff}([P],[\hat{Q}]) \xrightarrow{\mathscr{C}\,\mathrm{Diff}(\nabla,\mathrm{id}_{[\hat{Q}]})} H^n(\mathscr{C}\,\mathrm{Diff}\{[\kappa],[Q]\}) \\ &\quad = \mathscr{C}\,\mathrm{Diff}([\kappa],[\hat{Q}]), \end{aligned}$$

i.e., $\Delta \mapsto \Delta \circ \nabla$, $\Delta \in \mathscr{C}\,\mathrm{Diff}([P],[\hat{Q}])$. Let us check it is a monomorphism. Since $\mathscr{C}\,\mathrm{Diff}([\kappa],[\hat{Q}]) = \mathscr{C}\,\mathrm{Diff}([\kappa], \mathscr{F}(\mathscr{Y})) \otimes [Q]$, it suffices to prove the injectivity of the map

$$\mathscr{C}\,\mathrm{Diff}([P], \mathscr{F}(\mathscr{Y})) \xrightarrow{\mathscr{C}\,\mathrm{Diff}(\nabla,\mathrm{id}_{\mathscr{F}})} \mathscr{C}\,\mathrm{Diff}([\kappa], \mathscr{F}(\mathscr{Y})),$$

but the latter is obvious, since $\mathscr{C}\,\mathrm{Diff}(\nabla, \mathrm{id}_{\mathscr{F}}) = v_\nabla^{(1,0)}$.

Thus we have obtained the following result:

THEOREM. *Suppose* $K_0^{1,0}$ *possesses a direct complement in* $\mathscr{C}\,\mathrm{Diff}([\kappa], \mathscr{F}(\mathscr{Y}))$ *and* $\ker \mathscr{S}_\nabla = 0$ (*this will be so if* char ∇^* *is nowhere dense*). *Then, for* $p \geqslant 1$, *we have* $E_1^{p,q}(\mathscr{Y}) = H^q(K^{p,*}) = 0$ *if* $q \neq n-1$ *or* n. *Moreover* $E_1^{1,n-1}(\mathscr{Y}) = \ker \nabla^*$, $E_1^{1,n} = \mathrm{coker}\, \nabla^*$.

Proof. The first part of the theorem follows, in view of the above, from statement (2) of Theorem 10.11, and the second follows from (10.7.3) and statement (2) of Theorem 10.7. ∎

10.13. To conclude the section, let us indicate a method for calculating the term $E_2(\mathscr{Y})$. To do this we shall need the notion of homotopy in the category DE.

Suppose $\mathscr{O} \in Ob$ DE and $I = [0, 1]$. The algebra $\mathscr{F}(\mathscr{O} \times I)$ will be defined as the algebra of functions $f_t \in \mathscr{F}(\mathscr{O})$ which depend smoothly on the parameter $t \in I$. Put

$$\mathscr{F}_p(\mathscr{O} \times I) = \{f_t \in \mathscr{F}_p(\mathscr{O}) \mid \{\mathscr{F}_p(\mathscr{O})\} \text{ is the filtration in } \mathscr{F}(\mathscr{O})\}.$$

Thus we have defined an *FG*-category over $\mathscr{F}(\mathscr{O} \times I)$.

Consider I as an object of DE by supposing that for every representative object $\Phi(I)$ (see 6.4) we have

$$\mathscr{C}\Phi(I) = \{\varphi \in \Phi \mid \varphi(t) = 0, \forall t \in I\}.$$

In other words, the points $t \in I$ and only these points are integral manifolds

of the "Cartan distribution" on I. Suppose π_1, π_2 are the natural projections of $\mathscr{O} \times I$ on $\mathscr{O}$ and I, respectively. The operation $\mathscr{C}$ on $\mathscr{O} \times I$ shall be introduced under the assumption that the submodules $\mathscr{C}\Phi(\mathscr{O} \times I)$ of $\Phi(\mathscr{O} \times I)$ are generated by $\pi_1^*\mathscr{C}\Phi(\mathscr{O})$ and $\pi_2^*\mathscr{C}\Phi(I)$. Obviously the operation $\mathscr{C}$ introduced in this way transforms $\mathscr{O} \times I$ into an object of the category DE. Here the projections π_i, $i = 1, 2$, as well as the maps $\alpha_t : \mathscr{O} \to \mathscr{O} \times I$, $\alpha_t(x) = (x, t)$, $x \in \mathscr{O}$, are morphisms in DE.

Remark. If above we replace I by any object of the category DE, we obtain the definition of a direct product in DE.

DEFINITION. The morphisms $F_0, F_1 : \mathscr{O}_1 \to \mathscr{O}_2$ in DE are homotopic if there exists a morphism $F : \mathscr{O}_1 \times I \to \mathscr{O}_2$ such that $F_0 = F \circ \alpha_0$, $F = F_1 \circ \alpha_1$.

Since the term E_1 of the $\mathscr{C}$-spectral sequence is supplied with the infinitesimal Stokes' formula, while the morphisms in DE induce homomorphisms of the corresponding $\mathscr{C}$-spectral sequences, the same arguments as in the category of smooth manifolds establish the following result.

THEOREM. *If the morphism F_0 and $F_1 : \mathscr{O}_1 \to \mathscr{O}_2$ of the objects of the category* DE *are homotopic, then the homomorphisms*

$$F_i^* : E_2^{p,q}(\mathscr{O}_2) \to E_2^{p,q}(\mathscr{O}_1), \qquad i = 0, 1,$$

coincide.

This fact may be used to compute $E_2(\mathscr{O})$ along the same lines as in the computation of the de Rham cohomology of smooth manifolds. Let us mention a result in this direction, which is the analogue of Poincaré's lemma in the category DE.

Suppose $\mathscr{Y} \subset J^k(\pi)$, the bundle π being linear and the group of fibrewise homoteties of the bundle π being the symmetry group of the equation $\mathscr{Y}$. Such an equation $\mathscr{Y}$ will be called conic.

PROPOSITION. *If the equations $\mathscr{Y}$ is conic, then $E_2^{p,q}(\mathscr{Y}) = 0$, $p > 0$, and $E_2^{0,q}(\mathscr{Y}) = H^q(M)$, where M is the base of the bundle π.*

Proof. If σ is the zero section of the bundle π, then $j(\sigma)$ is a homotopy equivalence in DE. ∎

COROLLARY. *If the equation $\mathscr{Y}$ is conic, then the complexes* (10.7.4) *and* (10.7.5) *are acyclic in positive dimensions.*

11. Applications to Lagrangian Formalism with Constraints and the Theory of Conservation Laws

Here we have collected the consequences of results given in Sections 9 and 10. They are obtained by means of a more detailed description of the terms $E_1^{\varepsilon,n}$, $E_1^{\varepsilon,n-1}$ and the differentials $d_1^{\varepsilon,n}$, $d_1^{\varepsilon,n-1}$, where $\varepsilon = 0$ or 1.

11.1. In order to apply the $\mathscr{C}$-spectral sequence, we must have information concerning the group $H^*(\mathscr{Y}_\infty)$ to which it converges. Let us make some remarks in this connection, limiting ourselves, for the sake of simplicity, to the case $\mathscr{Y} \subset J^k(\pi)$. The general case may be reduced to this one by means of standard topological techniques and contains nothing knew in principle.

First let us note that the projection $\pi_{\infty,k} : \mathscr{Y}_\infty \to \mathscr{Y}$ is not surjective in general and, of course, need not be a fibre bundle. If we take some contradictory equation $\mathscr{Y}' \subset J^k(\pi)$ (this will be the case if $\mathscr{Y}' \subset \pi_k^{-1}(M')$, where M' is a submanifold in M and $\dim M' < \dim M$) and "glue it together" with a "nice" equation $\mathscr{Y}'' \subset J^k(M)$, we can obtain examples of this type which are as pathological as we wish. Thus the cohomologies of the "manifolds" $\mathscr{Y}$ and $\mathscr{Y}_\infty$ are not related to each other in any reasonable way in the general case. On the other hand, in the case when the natural projections $\mathscr{Y}^{s+1} \to \mathscr{Y}^s$ (see 6.2) are surjective, we can claim, under certain regularity conditions, that $\mathscr{Y}^{s+1}$ and $\mathscr{Y}^s$ have the same homotopy type. Then of course $H^*(\mathscr{Y}_\infty) = H^*(\mathscr{Y})$. The exact formulations of the appropriate assumptions can be found in [33, 34]. Here we mention a result due to Goldschmidt (see [33, 34]).

Proposition. *Suppose $\mathscr{Y} \subset J^k(\pi)$ is a subbundle of the bundle $\pi_k : J^k(\pi) \to M$, $\mathscr{Y}^1 \to \mathscr{Y}^0 = \mathscr{Y}$ is surjective and the symbol of the equation $\mathscr{Y}$ is 2-acyclic. Then for any $s \geqslant 0$ the projections $\mathscr{Y}^{s+1} \to \mathscr{Y}^s$ are affine bundles and therefore $H^*(\mathscr{Y}_\infty) = H^*(\mathscr{Y})$.*

In particular, if the equation $\mathscr{Y}$ is non-overdetermined and is a "good" subbundle of the bundle π_k, we have $H^*(\mathscr{Y}_\infty) = H^*(\mathscr{Y})$.

Corollary. *Suppose that for the equation $\mathscr{Y}$ we have the assumptions of Theorem* 10.11(2) *and of Proposition* 11.1. *Then $E_2^{0,n-1}(\mathscr{Y}) = H^{n-1}(\mathscr{Y})$.*

Proof. Indeed, Theorem 10.11 implies, in view of Corollary in 10.7, the relation written above. ∎

Thus, under our assumptions, we have the exact sequence

$$0 \to H^{n-1}(\mathscr{Y}) \to \bar{H}^{n-1}(\mathscr{Y}) = E_1^{0,n-1} \xrightarrow{d_1^{0,n-1}} E_1^{1,n-1} = \ker \nabla^*. \quad (11.1.1)$$

Recall that the group $\bar{H}^{n-1}(\mathscr{Y})$ was interpreted as the group of conservation laws for the equation $\mathscr{Y}$. Then the conservation law $\omega \in H^{n-1}(\mathscr{Y}) \subset \bar{H}^{n-1}(\mathscr{Y})$ will be called (rigid), since it is determined only by the topology of the equation $\mathscr{Y}$ and not by the specific traits of its solutions. In particular, if $V \subset \mathscr{N}$ is a solution of the equation $\mathscr{Y}$, then the cohomology class $j(V)^*(\omega)$ doesd not change under deformations of this solution.

DEFINITION. The quotient group $\bar{\bar{H}}^{n-1}(\mathscr{Y}) = \bar{H}^{n-1}(\mathscr{Y})/H^{n-1}(\mathscr{Y})$ (defined under our assumptions) will be called the group of proper conservation laws of equation $\mathscr{Y}$.

For the sequel, the class of equations described by the following definition will be important.

DEFINITION. (1) The equation $\mathscr{Y}$ which satisfies the assumptions of Theorem 10.11(2) and Proposition 11.1 will be called normal with respect to ∇.

(2) The equation $\mathscr{Y} = \{\varphi = 0\}$ which is normal with respect $\nabla = l_\varphi$ will be called normal.

THEOREM. *Suppose the equation $\mathscr{Y}$ is normal with respect to ∇. Then $\bar{\bar{H}}^{n-1}(\mathscr{Y}) \subset \ker \nabla^*$. If, moreover, $\bar{H}^n(\mathscr{Y}) \supset H^n(\mathscr{Y})$ (in particular $H^n(\mathscr{Y}) = 0$), we have $\bar{\bar{H}}^{n-1}(\mathscr{Y}) = \ker d_1^{1,n-1}$.*

Proof. The first part of the theorem follows from the exactness of 11.1. and the second from the fact that $E_2^{1,n-1}(\mathscr{Y}) = E_\infty^{1,n-1}(\mathscr{Y})$ (see Corollary 10.7). ∎

This gives an effective method for computing conservation laws or estimating their number. Indeed, an estimate from above is given by the computation of the kernel of the operator ∇^*, which in many cases is a relatively simple procedure. The operator $d_1^{1,n-1}$ also possesses an effective description (see 11.6). By using it, we may remove "superflous" elements from $\ker \nabla^*$ and obtain $\bar{\bar{H}}^{n-1}(\mathscr{Y})$. For example, this approach yields, without much difficulty, the conservation laws of such equations as the Korteveg–de Vries equation, the Burghers and Klein–Gordon equations and many others. One of the methods for carrying out such computations is described in [35].

Remark 1. It is useful to keep in mind that $\ker d_1^{1,n-1}/\bar{\bar{H}}^{n-1}(\mathscr{Y}) \subset H^n(\mathscr{Y})$, so that $\ker d_1^{1,n-1} = \bar{\bar{H}}^{n-1}(\mathscr{Y})$ up to certain easily verified topological conditions.

Remark 2. Suppose $\mathscr{Y} \subset J^k(\pi)$. Then if $H^n(\mathscr{Y}) = (\pi_k|_{\mathscr{Y}})^* H^n(M))$ we have $H^n(\mathscr{Y}) \subset \bar{H}^n(\mathscr{Y})$.

11.2. Now suppose the equation $\mathscr{Y} \subset J^k(\pi)$ itself is self- (anti-) adjoint. This means (see 9.8) that $\mathscr{Y} = \{\varphi = 0\}$ and $l_\varphi = \pm l_\varphi^*$. Also assume that it is normal. Then by (6.6.3), we have

$$\operatorname{Sym} \mathscr{Y} = \ker[l_\varphi] = \ker[l_\varphi^*] = E_1^{1,n-1}. \tag{11.2.1}$$

This gives us the following important result, which may be interpreted as the general inverse Noether theorem.

THEOREM. *Suppose the normal equation $\mathscr{Y} = \{\varphi = 0\} \subset J^k(\underline{\pi})$ is itself self- (anti-) adjoint. Then the operator $d_1^{0,n-1}$ embeds the group $\bar{\bar{H}}^{n-1}(\mathscr{Y})$ of proper conservation laws of the equation $\mathscr{Y}$ into the algebra* Sym $\mathscr{Y}$ *of its symmetries. In other words, to each conservation law of equation $\mathscr{Y}$ corresponds one of its symmetries and the kernel of this correspondence is $H^{n-1}(\mathscr{Y})$.*

COROLLARY. *If the normal equation $\mathscr{Y} \subset J^k(\pi)$ happens to be an Euler–Lagrange equation, while $H^{n-1}(E_\pi) = 0$ then the operator $d_1^{0,n-1}$ embeds the group of proper conservation laws of the equation $\mathscr{Y}$ into the algebra* Sym $\mathscr{Y}$ *of its symmetries.*

Proof. To prove the corollary, just compare Theorems 9.9 and 11.2. ∎

Remark 1. There is a great deal of literature devoted to the inverse Noether theorem for Euler–Lagrnage equations. The question of the validity of the Noether inverse theorem for an anti-adjoint equations has apparently never been raised.

Remark 2. Note that the results of this subsection and the previous ones are valid, as a rule, for the non-overdetermined equations that appear in geometry and mathematical physics, i.e., the normality assumptions are not too restrictive and, moreover, are easy to verify.

11.3. The substitution operation and the Lie derivative described in 9.10 enable us to introduce into the terms $E_1^{p,q}$ some original structures over the Lie algebra Sym $\mathscr{Y}$. Namely, let us put

$$\chi \downarrow e = d_1^{p-1,q}(\chi \lrcorner e), \qquad \chi \in \operatorname{Sym} \mathscr{Y}, \quad e \in E_1^{p,q},$$
$$\chi \uparrow e = \chi \lrcorner d_1^{p,q} e.$$

Then it follows from the infinitesimal Stokes formula (9.10.1) that

$$\chi\{e\} = \chi \downarrow e + \chi \uparrow e.$$

The standard rules which determine the relationship between the substitution of vector fields into differential forms and exterior differentiation enables one

to indicate axioms (A) which describe the general rules of behaviour of these operations. The above motivates the following definition.

DEFINITION. The module E over the Lie algebra S supplied with the operations $\downarrow : S \times E \to E$ and $\uparrow : S \times E \to E$ which satisfy the axioms (A) and for which we have $s(e) = s \downarrow e + s \uparrow e$, $s \in S$, $e \in E$ is called a $\updownarrow$-structure over S.

Note that the $D(M)$-modules $\Lambda^p(M)$ are naturally $\updownarrow$-structures over $D(M)$, as well as $E_1^{p,q}$ over Sym $\mathscr{Y}$.

Our aim here is merely to indicate the possible role of this structure in the study of the term E_1 of the $\mathscr{C}$-spectral sequence. Hence we shall not write out (A) explicitly.

Taking the composition of the operation $\uparrow$ and $\downarrow$ starting from a certain family of elements $\chi_i \in$ Sym $\mathscr{Y}$ and $e_j \in E_1^{p,q}$ we can successively obtain new elements in $E_1^{p,q}$. Since the explicit computation of the terms $E_1^{p,q}$ is possible only in very rare cases, the determination of algebraic conditions for the $\updownarrow$-structures under which, say, the operation of generating new elements described above yields an infinity of distinct elements is very important, especially for the terms $E_1^{0,n-1}$ and $E_1^{1,n-1}$, since they are directly related to the theory of conservation laws.

If the equations $\mathscr{Y}$ is normal and moreover self- or anti-adjoint then $E_1^{1,n-1} \approx$ Sym $\mathscr{Y}$ and the described procedure enales us to generate not only elements from $E_1^{1,n-1}$ (and therefore conservation laws) but elements from the algebra Sym $\mathscr{Y}$ as well. This in its turn opens up new possibilities for continuing the generating process etc. It should be noted that the adjoint action of the Lie algebra Sym $\mathscr{Y}$ onto itself does not coincide with its action on $E_1^{1,n-1}$ when $E_1^{1,n-1}$ and Sym $\mathscr{Y}$ are identified by means of this isomorphism (11.2.1).

Thus the study of $\updownarrow$-structures may turn out to be very important from the point of view of estimating the supply of symmetries and conservation laws of specific differential equations.

11.4. Let us indicate a description of the differential $d_1^{0,n-1}$ and the isomorphism $E_1^{1,n-1} = \ker \nabla^*$, since they are important for the theory of conservation laws. Among other things, this description would make clear why Theorem 11.2 is the converse of the classical Noether theorem.

We shall work within a single affine chart or, equivalently, we shall assume $\mathscr{Y}_\infty \subset J^\infty(\pi)$. The restriction to $\mathscr{Y}_\infty$ of various objects given on $J^\infty(\pi)$ are denoted, as in Section 10, by adding square brackets $[\cdot]$ to be the original notations. We shall also use the notations of 10.1 and shall put $e = \langle a \rangle$, when $e \in E_1^{p,q}$ and the representative d_0-cocycle is $a \in E_0^{p,q}$.

Identify E_0 (resp. $E_0(\mathscr{Y})$) with the bicomplex $\mathscr{C}\,\mathrm{Diff}^{\mathrm{alt}}\{\kappa\}$ (resp.

$\mathscr{C}$ Diff$^{\text{alt}}\{[\kappa]\}/K_0$) (see 10.3) and E_1 (resp. $E_1(\mathscr{Y})$) with the cohomologies of these complexes. Then if $e = \langle a \rangle$, $a \in E_0^{p,q}$, we have

$$d_1^{p,q}(e) = \langle l_{p,q}(a) \rangle. \tag{11.4.1}$$

If we take into consideration Proposition 10.3, we obtain

$$d_1^{p,q}([e]) = [\langle l_{p,q}(a) \rangle] \bmod K_0. \tag{11.4.2}$$

According to (9.4.1), $l_{0,q}(\omega) = l_\omega$, $\omega \in \Lambda_0$, therefore

$$d_1^{p,q}\langle[\omega]\rangle = [l_\omega] \bmod K_0. \tag{11.4.3}$$

This is the necessary formula.

Suppose further $e \in E_1^{1,n-1}(\mathscr{Y})$ and $e = \square \bmod K_0$, $\square \in \mathscr{C}$ Diff($[\kappa]$, $\bar{\Lambda}^{n-1}(\mathscr{Y})$), therefore $\bar{d} \circ \square = 0 \bmod K_0$, i.e., $\bar{d} \circ \square = \Delta \circ \nabla$ (recall that equation $\mathscr{Y}$ is regular with respect ∇). Since the isomorphism between $E_1^{1,n-1}(\mathscr{Y}) = H^{n-1}(K^{1,*})$ and $\ker \nabla^* = H^n(K_0^{1,*})$ is effected by the coboundary operator of the exact cohomology sequence corresponding to the exact sequence of complexes $0 \to K_0^{1,*} \to \widetilde{\mathscr{S}[\kappa]} \to K^{1,*} \to 0$ and $d_{1,q} = -\bar{d}$ (see 9.4) the element e is mapped by this isomorphism into the element $-\Delta^*(1) \in \ker \nabla^*$.

Conversely, suppose $\psi \in \ker \nabla^* \subset [\hat{P}] = \operatorname{Hom}_{\mathscr{F}(\mathscr{Y})}([P], \bar{\Lambda}^n(\mathscr{Y}_\infty))$. Then $\mu(\psi \circ \nabla) = \nabla^*(\psi) = 0$ (see 9.5), therefore $\psi \circ \nabla = \bar{d} \circ \square$ where, for example, by Green's formula, we have $\square(\chi) = Ж_\lambda(\psi \circ \nabla \circ \chi)$, so that the element $\psi \in \ker \nabla^*$ is mapped by the isomorphism under consideration into the element $e = -\square \bmod K_0 \in E_1^{1,n-1}(\mathscr{Y})$.

If $e = d_1^{0,n-1}\langle[\omega]\rangle$ and $\nabla = [l_\varphi]$ (see 10.5), in order to find the element from $\ker \nabla^*$ which corresponds to e, it is necessary, in view of (11.4.3) to find the operator Δ from the relation $\bar{d} \circ [l_\omega] = \Delta \circ [l_\varphi]$. But $\bar{d} \circ [l_\omega] = [l_{\bar{d}\omega}]$ and $\bar{d}\omega = 0$ on $\mathscr{Y}_\infty$. Hence $\bar{d}\omega = \square(\varphi)$, $\square \in \mathscr{C}$ Diff$(P, \bar{\Lambda}^n)$.

Since, moreover $[l_{\square(\varphi)}] = [\square] \circ [l_\varphi]$, we have $\Delta = [\square]$ and $\Delta^*(1) = [\square^*(1)]$. Thus we have proved the following statement, which describes the image of the operator $d_1^{0,n-1}$.

Proposition. *Suppose the equation $\mathscr{Y} = \{\varphi = 0\} \subset J^k(\pi)$ is normal. Identify $E_1^{1,n-1}(\mathscr{Y})$ with* $\ker[l_\varphi^*]$. *Then*

$$d_1^{0,n-1}(h) = [\square^*(1)], \qquad h = \langle[\omega]\rangle \in \bar{H}^{n-1}(\mathscr{Y}), \qquad \omega \in \bar{\Lambda}^{n-1},$$

where $\square \in \mathscr{C}$ Diff$(P, \bar{\Lambda}^n)$ *is an operator satisfying $\bar{d}\omega = -\square(\varphi)$.*

Corollary. *Suppose $f \in$* Sym $\mathscr{Y}$. *Then, under the assumptions of Proposition* 11.4, *we have*

$$\mathfrak{Z}_f\{h\} = (\mathfrak{Z}_f + \{\Delta^*\})(h), \qquad h \in \ker[l_\varphi^*], \qquad h = \{\tilde{h}\},$$

where the operator $\Delta \in \mathscr{C}\,\mathrm{Diff}(P, P)$ *satisfies* $\mathfrak{Z}_f(\varphi) = \Delta(\varphi)$. *In other words, the Lie derivative of the element* $h \in \ker[l_\varphi^*]$ *understood as an element of* $E_1^{1,n-1}$ *along the field* $\mathfrak{Z}_f$ *is equal to the "ordinary" derivative* $\mathfrak{Z}_f(h)$ *of this element plus* $[\Delta^*](h)$.

Proof. First note that the operator Δ exists, since $f \in \operatorname{Sym} \mathscr{Y}$ and the equation $\mathscr{Y}$ is normal. Let us calculate the operator $\tilde{\Box}$ such that $\bar{d}\mathfrak{Z}_f(\omega) = -\tilde{\Box}(\varphi)$. But $\bar{d}\mathfrak{Z}_f(\omega) = \mathfrak{Z}_f(\bar{d}\omega) = -\mathfrak{Z}_f(\Box(\varphi))$. Using the Green formula for $\Box(\varphi)$, we find

$$\begin{aligned}\mathfrak{Z}_f(\Box(\varphi)) &= \mathfrak{Z}_f(\langle \varphi, \Box^*(1)\rangle + \bar{d}\mathcal{K}_\lambda(\Box \circ \varphi))\\ &= \mathfrak{Z}_f(\langle \varphi, \tilde{h}\rangle) + \bar{d}\mathfrak{Z}_f(\mathcal{K}_\lambda(\Box \circ \varphi))\end{aligned}$$

since $\Box^*(1) = \tilde{h}$ and $\mathfrak{Z}_f \circ \bar{d} = \bar{d} \circ \mathfrak{Z}_f$. But

$$\begin{aligned}\mathfrak{Z}_f(\langle \varphi, \tilde{h}\rangle) &= \langle \mathfrak{Z}_f(\varphi), \tilde{h}\rangle + \langle \varphi, \mathfrak{Z}_f(\tilde{h})\rangle = \langle \Delta(\varphi), \tilde{h}\rangle + \langle \varphi, \mathfrak{Z}_f(\tilde{h})\rangle\\ &= (\tilde{h}^* \circ \Delta + \mathfrak{Z}_f(\tilde{h})^*)(\varphi).\end{aligned}$$

On the other hand, by using the formula for the infinitesimal transformation of $\mathcal{K}_\lambda$ along the field $\mathfrak{Z}_f$ (see 2.7) and taking into consideration (7.7.4), we obtain

$$\begin{aligned}\mathfrak{Z}_f(\mathcal{K}_\lambda(\Box \circ \varphi)) &= \mathfrak{Z}_f(\mathcal{K}_\lambda)(\Box \circ \varphi) + \mathcal{K}_\lambda([\mathfrak{Z}_f, \Box \circ \varphi])\\ &= \bar{d} \circ v(\Box \circ \varphi) + \mathcal{K}_\lambda(\Box \circ \mathfrak{Z}_f(\varphi) + \mathfrak{Z}_f(\Box) \circ \varphi)\\ &= \bar{d} \circ v(\Box \circ \varphi) + \mathcal{K}_\lambda(\Box \circ \Delta(\varphi) + \mathfrak{Z}_f(\Box) \circ \varphi).\end{aligned}$$

In view of (7.7.4), we see that the operator $\nabla : \psi \mapsto \mathfrak{Z}_f(\mathcal{K}_\lambda(\Box \circ \psi))$ is $\mathscr{C}$-differential. Thus

$$\mathfrak{Z}_f(\Box(\varphi)) = (\tilde{h}^* \circ \Delta + \mathfrak{Z}_f(\tilde{h})^* + \bar{d} \circ \nabla)(\varphi).$$

i.e., $\tilde{\Box} = \tilde{h} \circ \Delta + \mathfrak{Z}_f(\tilde{h})^* + \bar{d} \circ \nabla$ and $\mathfrak{Z}_f\{h\} = [\tilde{\Box}^*(1)] = \mathfrak{Z}_f(h) + [\Delta^*](h)$. ∎

11.5. Let us establish the relationship between Theorems 8.6 and 11.2 (to be more precise, Corollary 11.2), i.e., between the generalized "direct" and "inverse" Noether theorems. To do this, we give a restatement of Theorem 8.6 which is more satisfactory from the conceptual point of view.

DEFINITION. (1) The element $\chi \in \kappa(\pi)$ is said to be a symmetry of the Lagrangian $\Omega \in \bar{H}^n(J^\infty(\pi))$, if $\chi\{\Omega\} = 0$ on $\mathscr{Y}_\infty$, where $\mathscr{Y}$ is the Euler–Lagrange equation corresponding to Ω.

(2) If in the previous definition we have $\chi\{\Omega\} = 0$, then χ will be called a Noether symmetry of the Lagrangian Ω.

Denote the set of all symmetries of the Lagrangian Ω by $\operatorname{Sym}\Omega$ and the set of all its Noether symmetries by $\operatorname{sym}\Omega$. Obviously $\operatorname{sym}\Omega \subset \operatorname{Sym}\Omega$.

If χ is a symmetry of the Lagrangian Ω, while ρ is its density, then $Э_\chi(\rho) = \bar{d}\nu$ on $\mathscr{Y}_\infty$, $\nu \in \bar{\Lambda}^{n-1}(J^\infty(\Omega))$. Suppose the field $Y \in \mathscr{C}D(\mathscr{F}(\pi))$ satisfies $\nu = -Y \lrcorner \rho$. Then

$$(Э_\chi + Y)(\rho) = Э_\chi(\rho) + Y(\rho) = Э_\chi(\rho) - \bar{d}(Y \lrcorner \rho) = 0$$

on $\mathscr{Y}_\infty$. Therefore the $\mathscr{C}$-field $Э_\chi + Y = X$ is a symmetry of the density ρ (see 8.6) and, according to Theorem 8.6, the c-density $c_\lambda(\rho, X) = Ж_\lambda(l_\rho \circ \chi) - \nu$ corresponds to this symmetry. Thus we obtain the following map

$$\mathcal{N}t : \operatorname{Sym}\Omega \to \bar{H}^{n-1}(\mathscr{Y}), \qquad \chi \mapsto [\langle Ж_\lambda(l_\rho \circ \chi) - \nu \rangle],$$

which we shall call the Noether map. Let us clarify the dependence of this map on the choice of ρ and ν.

Suppose $\rho' = \rho + d\psi$. Then Green's formula implies

$$\bar{d}Ж_\lambda(l_{\bar{d}\psi} \circ \chi) = \bar{d}Ж_\lambda(\bar{d} \circ l_\psi \circ \chi) = \bar{d}(l_\psi(\chi))$$

and therefore $Ж_\lambda(l_{\bar{d}\psi} \circ \chi) = l_\psi(\chi) + \bar{d}\alpha$ if $\bar{H}^{n-1}(J^\infty(\pi)) = 0$ (or locally). On the other hand

$$[l_{\rho'}(\chi)] = [l_\rho(\chi)] + [l_{\bar{d}\psi}(\chi)] = [\bar{d}\nu] + [\bar{d}l_\psi(\chi)]$$

so that for the element "ν" corresponding to ρ' we can take $\nu + l_\psi(\chi)$. This means that the Noether map does not depend on the choice of ρ locally but is in general multivalued and the "degree" of multivaluedness coincides with the image of the homomorphism $\bar{H}^{n-1}(J^\infty(\pi)) \to \bar{H}^{n-1}(\mathscr{Y})$. The arbitrariness in the choice of ν clearly leads to the same multivaluedness. Thus we have the following;

PROPOSITION. *The Noether map is well defined up to the image of the natural homomorphism*

$$H^{n-1}(J^\infty(\pi)) = \bar{H}^{n-1}(J^\infty(\pi)) \to \bar{H}^{n-1}(\mathscr{Y}).$$

If χ is a Noether symmetry, then $Э_\chi(\rho) = \bar{d}\nu$ and, according to Green's formula

$$\begin{aligned} 0 = l_\rho(\chi) - \bar{d}\nu &= \bar{d}\{Ж_\lambda(l_\rho \circ \chi) - \nu\} + \langle \mathscr{E}(\Omega), \chi \rangle \\ &= \bar{d}c_\lambda(\rho, X) + \langle \mathscr{E}(\Omega), \chi \rangle. \end{aligned}$$

According to Proposition 11.4, the element from $\ker[l_\varphi^*] = \ker[l_\varphi]$, where $\varphi = \mathscr{E}(\Omega)$, corresponding to the conservation law $h = \langle [\omega] \rangle$, $\omega = c_\lambda(\rho, X)$

equals $[\square^*(1)]$, where $\bar{d}\omega = -\square(\varphi)$. But $\bar{d}\omega = -\langle \mathscr{E}(\Omega), \chi \rangle = -\langle \varphi, \chi \rangle$ so that for the operator $\square$, we may take the element χ understood as an element of $\mathrm{Hom}(\hat{\kappa}, \bar{\Lambda}^n) \subset \mathscr{C}\,\mathrm{Diff}(\kappa, \bar{\Lambda}^n)$. Thus, if we identify $E_1^{1,n-1}$ with $\ker[l_\varphi^*] = \ker[l_\varphi] = \mathrm{Sym}\,\mathscr{Y}$, we get

$$d_1^{0,n-1}\langle [c_\lambda(\rho, X)] \rangle = d_1^{0,n-1}(\mathscr{N}t(\chi)) = \chi.$$

Thus we have proved the following:

THEOREM. *Suppose the normal equation $\mathscr{Y} \subset J^k(\pi)$ is an Euler–Lagrange equation corresponding to the Lagrangian $\Omega \in \bar{H}^n(J^\infty(\pi))$. Then the Noether map considered on set of Noether symmetries of the Lagrangian has the differential $d_1^{0,n-1}$ for its inverse map.*

Remark 1. It follows from these arguments that the theorem breaks down in general for non-Noether symmetries and the appropriate additional summand equals $[\square^*(1)]$, where $l_\rho(\chi) - \bar{d}v = \square(\varphi)$.

Remark 2. The search for Noether symmetries of a Lagrangian is a constructive procedure, since, in view of the properties of the variation complex described in 9.8 it reduces, under appropriate topological conditions, or locally, to solving the equation $\mathscr{E}(3_\chi(\rho)) = 0$, $\Omega = \langle \rho \rangle$. In the general case, we need information on zero dimensional cohomology of the complex (10.7.5) or, which is the same, on the term $E_2^{0,n}(\mathscr{Y})$.

11.6. In view of the particular importance of the operator $d_1^{1,n-1}$ due to the relation $\ker d_1^{1,n-1} = \bar{\bar{H}}^{n-1}(\mathscr{Y})$ (Theorem 11.1), let us describe it in greater detail. To do this we shall need some supplementary information on the term $E_1^{2,n-1}(\mathscr{Y})$. In this subsection we shall assume that we have the conditions of Theorem 10.11(2).

Using (10.6.2) for $p = 2$ and noticing that the kernel of the alternation map

$$\mathscr{C}\,\mathrm{Diff}\{[\kappa]; [\kappa]\} \to \mathscr{C}\,\mathrm{Diff}_{(2)}^{\mathrm{alt}}\{[\kappa]\}$$

is $\mathscr{C}\,\mathrm{Diff}_{(2)}^{\mathrm{sym}}\{[\kappa]\}$, it is easy to verify that $\ker v_\nabla^{(2)}$ coincides with the image of the complex $\mathscr{C}\,\mathrm{Diff}_{(2)}^{\mathrm{sym}}\{[P]\}$ under the composition of maps

$$\mathscr{C}\,\mathrm{Diff}_{(2)}^{\mathrm{sym}}\{[P]\} \hookrightarrow \mathscr{C}\,\mathrm{Diff}_{(1)}\{[P]; [P]\} \xrightarrow{\mathscr{C}\mathrm{Diff}(\nabla;\mathrm{id})} \mathscr{C}\,\mathrm{Diff}_{(1)}\{[\kappa]; [P]\}$$

which is obviously monomorphic under our assumptions.

Taking into consideration the fact that $H^{n-1}(K_0^{p,*}) = 0$ in our case and using the isomorphism $\ker v_\nabla^{(2)} = \mathscr{C}\,\mathrm{Diff}_{(2)}^{\mathrm{sym}}\{[P]\}$ and (10.7.3) for $p = 2$, we get

$$0 \to \mathscr{C}\,\mathrm{Diff}^{\mathrm{sym}}([P], [\hat{P}]) \to \ker \nabla_{(2)} \to H^{n-1}(K^{2,*}) = E_1^{2,n-1} \to 0, \quad (11.6.1)$$

where $\mathscr{C}\,\mathrm{Diff}^{\mathrm{sym}}(Q, \hat{Q})$ denotes the set of all self-adjoint operators acting from Q into $\hat{Q}$ and, according to 9.6,

$$H^n(\ker v_\nabla^{(2)}) = H^n(\mathscr{C}\,\mathrm{Diff}_{(2)}^{\mathrm{sym}}\{[P]\}) = \mathscr{C}\,\mathrm{Diff}^{\mathrm{sym}}([P], [\hat{P}]).$$

Since $\ker \nabla_{(2)} \subset H^n(\mathscr{C}\,\mathrm{Diff}_{(1)}\{[\kappa]; [P]\}) = \mathscr{C}\,\mathrm{Diff}([\kappa], [\hat{P}])$ it follows from (11.6.1) that the group $E_1^{2,n-1}$ may be viewed as the subgroup of $\mathscr{C}\,\mathrm{Diff}([\kappa], [\hat{P}])/\Theta$, where Θ is the image of the inclusion

$$\mathscr{C}\,\mathrm{Diff}^{\mathrm{sym}}([P], [\hat{P}]) \hookrightarrow \mathscr{C}\,\mathrm{Diff}([\kappa], [\hat{P}]), \square \mapsto \square \circ \nabla.$$

PROPOSITION. *Suppose the equation $\mathscr{Y} = \{\varphi = 0\} \subset J^k(\pi)$ is normal and $[f] \in \ker[l_\varphi^*], f \in \hat{P}$. Then, identifying $E_1^{1,n-1}$ with $\ker[l_\varphi^*]$, we have*

$$d_1^{1,n-1}([f]) = [l_f + \Delta^*] \bmod \Theta,$$

where

$$l_\varphi^*(f) = \Delta(\varphi), \Delta \in \mathscr{C}\,\mathrm{Diff}(\hat{P}, \hat{\kappa}).$$

Proof. Notice at once that $l_\varphi^*(f) = 0$ on $\mathscr{Y}_\infty$. Since the equation $\mathscr{Y}$ is normal and therefore regular, there exists an operator Δ such that $l_\varphi^*(f) = \Delta(\varphi)$.

Let us write out the Green formula for the operator $f \circ l_\varphi \in \mathscr{C}\,\mathrm{Diff}(\kappa, \bar{\Lambda}^n)$ in the following form

$$(f \circ l_\varphi)(\chi) - \langle l_\varphi^*(f), \chi\rangle = \bar{d}(\square(\chi)), \qquad \chi \in \kappa(\pi). \tag{11.6.2}$$

Then, under the isomorphism $\ker[l_\varphi^*] \to E_1^{1,n-1}$, the element $[f]$ corresponds to $\langle[\square]\rangle \in E_1^{1,n-1}$ and by (11.4.2),

$$d_1^{1,n-1}\langle[\square]\rangle = [\langle l_{1,n-1}(\square)\rangle] = [l_{1,n-1}(\square)] \bmod K_0.$$

The image of the element $d_1^{1,n-1}\langle[\square]\rangle$ under the inclusion $E_1^{2,n-1} = H^{n-1}(K^{2,*}) \to H^n(K_0^{2,*})$ (see (7.7.2)) equals $\langle e\rangle$, where

$$e = \bar{d}_{2,n-1} \circ [l_{1,n-1}(\square)] = [l_{1,n}(\bar{d} \circ \square)] = [l_{1,n}(f \circ l_\varphi - l_\varphi^*(f))],$$

$l_\varphi^*(f) \in \hat{\kappa}$ is understood as an operator from κ to $\bar{\Lambda}^n$ and the last of the equalities follows from (11.6.2). But (9.4.1) implies $l_{1,n}(f \circ l_\varphi) = (l_f, l_\varphi) - (l_\varphi, l_f)$, where

$$(l_f, l_\varphi), (l_\varphi, l_f) \in \mathscr{C}\,\mathrm{Diff}_{*,2}(\kappa; \bar{\Lambda}^n), (l_f, l_\varphi)(\chi_1, \chi_2) = \langle l_f(\chi_1), l_\varphi(\chi_2)\rangle$$

(and similarly for (l_φ, l_f)) while the brackets $\langle\cdot\rangle$ in the right-hand side in the last relation denote the natural pairing $\hat{P} \times P \to \bar{\Lambda}^n$.

Applying (9.4.1) once again, we see that

$$l_{1,n}(h) = (l_h, \mathrm{id}_\kappa) - (\mathrm{id}_\kappa, l_h),\ h \in \hat{\kappa} \subset \mathscr{C}\ \mathrm{Diff}(\kappa, \bar{\Lambda}^n).$$

If we also have $h = l_\varphi^*(f) = \Delta(f)$, then $l_h = l_{\Delta(f)} = \Delta \circ l_f + \varphi^* \circ \Delta'$, where $\Delta' \in \mathscr{C}\ \mathrm{Diff}(\kappa, \mathrm{Hom}(P, \hat{\kappa}))$ is an operator.

Bringing all the above together, we find

$$\begin{aligned} e &= [l_{1,n}(f \circ l_\varphi - l_\varphi^*(f))] \\ &= [(l_f, l_\varphi) - (l_\varphi, l_f) - (\Delta \circ l_\varphi, \mathrm{id}_\kappa) + (\mathrm{id}_\kappa, \Delta \circ l_\varphi)]. \end{aligned}$$

Hence the inverse image $\bar{e}$ of the element e under the natural projection

$$\mathscr{C}\ \mathrm{Diff}_{(1)}\{[\kappa]; [P]\} \to K_0^{2,*} \subset \mathscr{C}\ \mathrm{Diff}_{(2)}^{\mathrm{alt}}\{[\kappa]\},\ \partial \mapsto (\partial \circ [l_\varphi])_{\mathrm{alt}},$$

obviously equals $(l_f, \mathrm{id}_P) - (\mathrm{id}_\kappa, \Delta)$. This implies that the corresponding cohomology class $\langle e \rangle \in \mathscr{C}\ \mathrm{Diff}([\kappa], [\hat{P}])$ equals (see 9.5) $\mu(e) = l_f + \Delta^*$. ∎

Remark. It is useful to keep in mind that Θ consists of all operators of the form $\square \circ [l_\varphi]$, where $\square = \square^* \in \mathscr{C}\ \mathrm{Diff}([P], [\hat{P}])$.

COROLLARY. *If under the assumptions of the proposition proved above, we have $f = (\pi_\infty|_{\mathscr{Y}_\infty})^*(g)$, then $d_1^{1,n-1}([f]) = 0$, i.e. $[f]$ determines a conservation law of the equation $\mathscr{Y}$.*

Proof. Indeed, in this case $l_f = 0$, $\Delta = 0$.

This corrolary shows that for a normal linear equation every solution of the adjoint equation determines a conservation law and such conservation laws are independent (see 4.4).

11.7. The Lagrangian formalism with constraints. The fact that the differential $d_1^{0,n}$ and the Euler operator coincide, established in 9.8 in the case of the Lagrangian formalism without constraints, being quite satisfactory from the functorial point of view, gives grounds to hope that in the formalism with constraints this fact will remain valid. Below we will show that this is indeed the case. To be more precise, we will consider the case when the "quantity" which is subjected to variation satisfies the "constraint equation" which is a differential equation $\mathscr{Y} \subset N_m^k$. This means, in the notation of 8.1, that the domain B', determined by such a variational problem is an open set in $\mathscr{Y}_\infty$.

Recall that, according to 8.1 and 8.2, the density of a Lagrangian should be understood as a form $\omega \in \bar{\Lambda}^n(B, \partial B)$ and a Lagrangian as the cohomology class $\mathscr{L} = \langle \omega \rangle \in \bar{H}^n(B, \partial B)$.

In contrast with Section 8, where all the arguments were based on the formula for the first variation, i.e., the variational problems were considered

in "naive" form, we shall now need the Stokes infinitismal formula (9.10.1) reduced to $L \in \mathscr{M}^n_{\mathrm{adm}}(N)$ (resp. $\sigma \in \Gamma_{\mathrm{adm}}(\pi)$). The special character of variational problems with constraints manifests itself in that the infinitismal variation of the solution L of the constraint equation, i.e., the "tangent vector" to the "solution space" of the constraint equation at the "point L" as a rule is not a value at L of a global vector field, i.e., an element of the algebra Sym $\mathscr{Y}$ on this "solution space." Therefore we must begin by specifying the notion of such a "tangent vector." To do this, assuming that $\mathscr{Y} = \{\varphi = 0\}$, $\varphi \in P$, or within a certain affine chart, consider the restriction $l_{\varphi,\sigma}$ of the operator l_φ to the section $\sigma \in \Gamma_{\mathrm{loc}}(\pi)$

$$l_{\varphi,\sigma} \in \mathrm{Diff}(\Gamma(\pi), P_\sigma), \qquad P_\sigma = j(\sigma)^* \, (P).$$

If $P = \mathscr{F}(\pi, \xi)$ (see 6.6), we have $P_\sigma = \Gamma(\xi)$ and $l_{\varphi,\sigma} = j(\sigma)^* \circ l_\varphi \circ \pi_\infty^*$. If in this case σ is a solution of the equation $\mathscr{Y}$, then every solution χ_σ of the equation $l_{\varphi,\sigma}(\chi_\sigma) = 0$ may be interpreted as a tangent vector at the "point" σ (or as an infinitismal variation of the solution σ) to the solution space of the equation $\mathscr{Y}$. In the general case, the element $\chi_L \in j(L)^* \, [\kappa]$, $\kappa = \kappa(N^\infty_m)$ will be called a "tangent vector" at L to the solution space of equation $\mathscr{Y}$ if χ_L is the solution, within an arbitrary affine chart of the equation $l_{\varphi,\sigma} = 0$, where the section σ is such that the graphs of the maps $j(\sigma)$ and $j(L)$ coincide within the chart under consideration. The set of all such "vectors" will be denoted by $\tau_L(\mathscr{Y})$ or $\tau_\sigma(\mathscr{Y})$ if $\mathscr{Y} \subset J^\infty(\pi)$.

The "infinitesmal Stokes formula" which we need corresponds to the value of formula (9.10.1) in the case $p = 0$ on L (resp. σ) and is of the form

$$\chi_L\{e\} = \chi_L \rfloor d_1^{0,q}(e), \qquad \chi_L \in \tau_L(\mathscr{Y}),$$

respectively,

$$\chi_\sigma\{e\} = \chi_\sigma \rfloor d_1^{0,q}(e), \qquad \chi_\sigma \in \tau_\sigma(\mathscr{Y}). \tag{11.7.1}$$

The elements of both sides of these formulas belong to $H^q(L)$ (resp. $H^q(\mathscr{U})$, where $\mathscr{U}$ is the domain of σ), and are defined in an obvious way. For example, $\chi_\sigma\{e\} = \langle l_{\rho,\sigma}(\chi_\sigma)\rangle$, where $e = \langle[\rho]\rangle$, $\rho \in \bar{\Lambda}^q(J^\infty(\pi))$. The proof of formula (11.7.1) is the same as that of (9.10.1) and is therefore omitted.

When we study a variational problem, it is possible to take as an "infinitesmal variation" of the manifold $L \in \mathscr{M}^n_{\mathrm{adm}}(N)$ (resp. $\sigma \in \Gamma_{\mathrm{adm}}(\pi)$ such elements $\chi_L \in \tau_L(\mathscr{Y})$ (resp. $\chi_\sigma \in \tau_\sigma(\mathscr{Y})$) which are in agreement with boundary conditions or some other conditions appearing in the problem. In other words, in $\tau_L(\mathscr{Y})$ (resp. $\tau_\sigma(\mathscr{Y})$) it is necessary to choose the sub-space of "tangent vectors" to $\mathscr{M}^n_{\mathrm{adm}}(N)$ (resp. $\Gamma_{\mathrm{adm}}(\pi)$). Such subspaces will be denoted by Adm(L) (resp. Adm(σ)). Their exact description depends on the type of the problem under consideration and, as a rule, does not involve any

difficulties. For example, the boundary conditions of the original variational problem become, in an obvious way, boundary conditions which must be satisfied by the solutions of the equation $l_{\varphi,\sigma} = 0$ in order that they belong to $\mathrm{Adm}(\sigma)$.

To deduce the Euler–Lagrange equations it remains to notice that the velocity of change of the functional under consideration under the infinitismal variation $\chi_L \in \mathrm{Adm}(L)$ (resp. $\chi_\sigma \in \mathrm{Adm}(\sigma)$) equals $\chi_L(\mathscr{L})$ (resp. $\chi_\sigma(\mathscr{L})$) by definition.

DEFINITION. The variational problem is said to be regular, if the condition $\chi_L \lrcorner\, d_1^{0,n}(\mathscr{L}) = 0$ (resp. $\chi_\sigma \lrcorner\, d_1^{0,n}(\mathscr{L}) = 0$) $\forall \chi_L \in \mathrm{Adm}(L)$ (resp. $\forall \sigma \in \mathrm{Adm}(\sigma)$) implies $j(L)^*\,(d_1^{0,n}(\mathscr{L})) = 0$ (resp. $j(\sigma)^*\,(d_1^{0,n}(\mathscr{L}) = 0)$. It should be noted that, strictly speaking, the operators $d_1^{1,n}$ appearing in this definition should be viewed as differentials of the $\mathscr{C}$-spectral sequence of the pair $(B, \partial B)$.

Having this definition in mind, as well as what we explained above, and formula (11.7.1), we immediately reach the following conclusion:

THEOREM. *A necessary condition for the extremality of the manifold L (resp. the section σ) in a regular variational problem is the relation $j(L)^*$ $(d_1^{0,n}(\mathscr{L})) = 0$ (resp. $j(\sigma)^*$ $({}_1^{0,n}(\mathscr{L})) = 0$).*

In other words, equation $d_1^{0,n}(\mathscr{L}) = 0$ is the analogue of the Euler–Lagrange equation of the problem with constraints. It should, however, be noted that the terms $E_1^{1,n}(\mathscr{Y}) \ni d_1^{0,n}(\mathscr{L})$ are not modules over the main algebra $\mathscr{F}(\mathscr{Y})$. It is therefore necessary to give meaning to expressions such as $j(L)^*\,(e)$, $e \in E_1^{1,n}(\mathscr{Y})$, which appear above. This is done in the following subsection. There we show that the Theorem 11.7 may be understood as a general result of the Lagrange multiplier theorem type.

11.8. The terms $E_0^{p,q} = E_0^{p,q}(\mathscr{O})$ of the $\mathscr{C}$-spectral sequence of some object $\mathscr{O}$ of the category DE are obviously $\mathscr{F}(\mathscr{O})$-modules, where the differentials $d_0^{p,q}$, as seen from (9.2.1), are $\mathscr{C}$-differential operators. Therefore the term $\{E_0, d_0\}$ of the $\mathscr{C}$-spectral sequence possesses a restriction to the graph of the map $j(L)$ (resp. $j(\sigma)$). To be more precise, we must consider the restrictions $E_{0,L}^{p,q} = j(L)^*\,(E_0^{p,q})$ (resp. $E_{0,\sigma}^{p,q} = j(\sigma)^*\,(E_0^{p,q})$) on which the operators $d_0^{p,q}$, in view of the $\mathscr{C}$-differentiality, generate operators $d_{0,L}^{p,q}$: $E_{0,L}^{p,q} \to E_{0,L}^{p,q+1}$ (resp. $d_{0,\sigma}^{p,q} : E_{0,\sigma}^{p,q} \to E_{0,\sigma}^{p,q+1}$). Obviously $d_{0,L}^{p,q+1} \circ d_{0,L}^{p,q} = 0$ (resp. $d_{0,\sigma}^{p,q+1} \circ d_{0,\sigma}^{p,q} = 0$), so that $\{E_{0,L}^{p,q}, d_{0,L}^{p,q}\}_{p=\mathrm{const}}$ (resp. $\{E_{0,\sigma}^{p,q}, d_{0,\sigma}^{p,q}\}_{p=\mathrm{const}}$ are complexes whose cohomology in the term $E_{0,L}^{p,q}$ (resp. $E_{0,\sigma}^{p,q}$) will be denoted by $E_{1,L}^{p,q}$ (resp. $E_{0,\sigma}^{p,q}$). Since these complexes are quotient complexes of the complex $\{E_0^{p,q}, d_0^{p,q}\}_{p=\mathrm{const}}$ a natural map of the cohomology is defined $E_1^{p,q} \to E_{1,L}^{p,q}$ (resp. $E_1^{p,q} \to E_{1,\sigma}^{p,q}$). These maps will be denoted by $j(L)^*$ (resp.

$j(\sigma)^*$). It is in this sense that this notation in is used in the definitions and the theorem of the previous subsection.

Now let us clarify the meaning of the equation $d_1^{0,n}(\mathscr{L})=0$. To do this, we assume that the constraint equation $\mathscr{Y}$ is regular with respect to the operator ∇ and we will use the relation $E_1^{1,n}=\operatorname{coker}\nabla^*$ (Theorem 10.7), and the natural restriction maps $E_r^{p,q}(N_m^\infty)\to E_r^{p,q}(B)$, if $B\subset\mathscr{Y}_\infty\subset N_m^\infty$ (resp. $E_r^{p,q}(J^\infty(\pi))\to E_r^{p,q}(B)$, if $B\subset\mathscr{Y}_\infty\subset J^\infty(\pi)$). This implies in particular that if $\mathscr{L}=\langle\omega\rangle$ and $\omega=[\rho]$, $\rho\in\bar{\Lambda}^n(N_m^\infty)$ (resp. $\rho\in\bar{\Lambda}^n(J^\infty(\pi))$), then

$$d_1^{0,n}(\mathscr{L})=\{[d_1^{0,n}(\langle\rho\rangle)] \bmod \operatorname{im}\nabla^*\}\in[\hat{\kappa}].$$

In particular, if $\mathscr{Y}=\{\varphi=0\}\subset J^k(\pi)$, then

$$d_1^{0,n}(\mathscr{L})=[l_\rho^*(1)] \bmod \operatorname{im}\nabla^*.$$

Therefore if by ∇_σ we denote the restriction of the operator ∇ to the section σ, then the relation $j(\sigma)^*\,d_1^{0,n}(\mathscr{L})=0$ is equivalent to $j(\sigma)^*\,(l_\rho^*(1))+\nabla_\sigma^*(\hat{p}_\sigma)=0$, where $\hat{p}_\sigma\in j(\sigma)^*\,(\hat{P})$, or $j(\sigma)^*\,([l_\rho^*(1)]+\nabla^*([\hat{p}]))=0$, where $\hat{p}\in\hat{P}$, $\hat{p}_\sigma=j(\sigma)^*\,(\hat{p})$. If $\mathscr{Y}=\{\varphi=0\}$ and $\nabla=[l_\varphi]$, then

$$[l_{(\hat{p},\varphi)}]=[(l_{\hat{p}},\varphi)+(\hat{p},l_\varphi)]=[(\hat{p},l_\varphi)]=[\hat{p}^*\circ l_\varphi].$$

Therefore $\nabla^*([\hat{p}])=[l_\varphi^*(\hat{p})]=[(\hat{p}^*\circ l_\varphi)^*\,(1)]=[l_{(\hat{p},\varphi)}^*(1)]$. Thus the extremality condition of the section σ takes the form

$$j(\sigma)^*\,([l_\rho^*(1)]+[l_{(\hat{p},\varphi)}^*(1)])=j(\sigma)^*\,(l_{\rho+(\hat{p},\varphi)}^*(1))=0.$$

Since $l_{\rho+(\hat{p},\varphi)}^*(1)=\mathscr{E}(\bar{\mathscr{L}})$, where $\bar{\mathscr{L}}=\langle\rho+(\hat{p},\varphi)\rangle$, the relation obtained above proves the following

THEOREM. *Any extremal σ of the given variational problem is simultaneously an extremal of the "free" Lagrangian $\bar{\mathscr{L}}=\bar{\mathscr{L}}(\hat{p})$ for some $\hat{p}\in\hat{P}$.*

This is the interpretation of Theorem 11.7 as a "Lagrange multiplier theorem." Indeed the role of the Lagrange λ-multiplier is played by the element $\hat{p}\in\hat{P}$.

The method for the further solution of the given variational problem consists in the following. We must consider the system of equations

$$\varphi=0,\qquad \mathscr{E}(\bar{\mathscr{L}}(\hat{p}))=0. \tag{11.8.1}$$

This system is in general inconsistent for an arbitrary $\hat{p}$. Therefore we must write out its consistency conditions and, solving them, find admissible an "Lagrange multiplier $\hat{p}$." Further the extremals of the given problem are

searched for as solutions of the system (11.8.1) for admissible $\hat{p}$ which satisfy the appropriate additional conditions (e.g., boundary conditions).

11.9. The results obtained above enable us to state the inverse problem of the calculus of variations for problems with constraints and indicate solvability conditions for them. Namely, the nonlinear operator given by the element $\mathscr{E}(\mathscr{L})$, where $\mathscr{L}$ is the same as in the previous two subsections, will be called the Euler–Lagrange operator for problems with constraints. We then have

THEOREM. *For an element $e \in E_1^{1,n}(B)$ to be an Euler–Lagrange operator of some variational problem, it is necessary that $d_1^{1,n}(e) = 0$. This condition is sufficient if the constraint equation is conic.*

Proof. This follows from the fact that the complex (10.7.5) is exact and from Corollary 10.13. ∎

Clearly this fact is useful to analize a more delicate question: is the given equation $e = 0$, $e \in E_1^{1,n}$, an Euler–Lagrange equation (compare with 9.9)?

The fact that the complex (10.7.5) is exact for conic equations also yields the following results.

PROPOSITION. *Suppose $\mathscr{L}$ is the Lagrangian in the problem without constraints and $\mathscr{Y} = \{\mathscr{E}(\mathscr{L}) = 0\}$. If the equation $\mathscr{Y}$ is normal and conic, while $H^n(\mathscr{Y}) = 0$ then $\mathscr{L}|_{\mathscr{Y}_\infty} = 0$.*

COROLLARY. *"The generalized Schwartz formula" (see the Introduction) exists for extremals of Lagrangians possessing a normal conic Euler–Lagrange equation.*

Proof. Indeed if $\mathscr{L} = \langle \omega \rangle$, then the relation $\mathscr{L}|_{\mathscr{Y}_\infty} = 0$ means that $\omega = \bar{d}\theta$ on $\mathscr{Y}_\infty$, i.e., if $\sigma : \mathscr{U}_\sigma \to E_\pi$ is an extremal, then $\int_{\mathscr{U}_\sigma} \sigma^*(\omega) = \int_{\partial \mathscr{U}_\sigma} \sigma^*(\theta)$. ∎

Note that the generalized Schwartz formula exists if and only if $\mathscr{L}|_{\mathscr{Y}_\infty} = 0$, where $\mathscr{Y} = \{\mathscr{E}(\mathscr{L}) = 0\}$. Moreover the form θ can be found directly by means of the infinitismal Stokes formula in $E_1(\mathscr{Y})$. For example, if the equation is conic, then the answer is given by formula (9.12.3).

12. CONCLUDING REMARKS

12.1. The framework of the category DE used in this article may be considerably widened. The generalized category DE (further denoted by GDE) contains among its objects, for instance, integro-differential equations and Wahlquist–Estabrook prolongated structures [40], so that its objects

may in general be interpreted as prolongated "generalized" differential equations. The study of GDE is undoubtedly very important from various points of view and, above all, because various generalized differential equations (e.g., integro-differential ones) costantly appear in the problems of theoretical and mathematical physics. The theory of nonlocal symmetries and conservation laws, which is just developing (see [35]), is important because, among other things, apparently almost any equation possesses an infinite supply of symmetries and conservation laws of this type, and this has its origin in the framework of GDE.

In connection with the above, it is useful to note that the principal constructions (e.g., the $\mathscr{C}$-spectral sequence) of this paper can automatically be introduced into the framework of GDE. Say, if $\mathscr{O}$ is a n-dimensional object of the category GDE and $\mathscr{L} \in H^n(\mathscr{O})$ is a Lagrangian, then $d_1^{0,n}(\mathscr{L})$ is the corresponding Euler–Lagrage equation.

12.2. Despite the fact that the objects of DE (and GDE) are infinitely prolongated equations, the final product of the Lagrangian formalism—the Euler–Lagrange equation—is not infinitely prolongated. This seems unsatisfactorily from the logical point of view. This defect of the theory may be eliminated if we construct the $\mathscr{C}$-spectral sequence by using, instead of the ordinary de Rham complex, its "higher" analogue, corresponding to the sequence $\sigma = (\infty, \infty, \ldots)$ (see [3]). The "higher" $\mathscr{C}$-spectral sequence which thus arises may turn out to be interesting from other points of view as well.

12.3. Suppose $\mathscr{G}$ is a Lie algebra of $\mathscr{C}$-fields on $\mathscr{O} \in Ob\,\mathrm{DE}$ (or GDE). Then we can consider, on $\mathscr{O}$, the de Rham complex $\Lambda^*_{\mathscr{G}}$ invariant with respect $\mathscr{G}$. By filtering this complex by powers of the ideal $\mathscr{C}\Lambda^*_{\mathscr{G}} = \Lambda^*_{\mathscr{G}} \cap \mathscr{C}\Lambda^*$, we will obtain a $\mathscr{G}$-invariant version of the $\mathscr{C}$-spectral sequence. T. Tsujishita noted the great interest (which was obvious a priori) of studying such spectral sequences, intepreted some known results and some of their terms (see [30]). For example, the results of Gelfand–Fuks [41] may be viewed as the computation of the $\mathscr{C}$-spectral sequence in the following situation: $\pi : E_\pi \to M$ is the n-frame bundle over the n-dimensional manifold M and $\mathscr{G}$ is the Lie algebra of the diffeomorphism group of the manifold M. Some of the results of Gilkey [42] may be interpreted, in a similar way as well as the theory of secondary characteristic classes of foliations etc.

12.4. The study of the $\mathscr{C}$-spectral sequence of equations such as the integrability equation of G-structures as well as the $\mathscr{G}$-analogues of this spectral sequence for appropriate $\mathscr{G}$ appear to be very interesting. Such aspects as various types of characteristic classes, Bott obstructions to integrability, characteristic classes of deformations, acquire a natural interpretation in terms of this spectral sequence. Meaningful observations in this

direction may be found in [30]. The appearance of complexes of the Spencer type in the computation of the term E_1 in Sections 9, and 10 appear to be far from accidental, if we recall that this type of complex was first introduced in theory of deformations of structures [43].

12.5. The study of the relative $\mathscr{C}$-spectral sequence of the pair $(B, \partial B)$, where B is a domain in $\mathscr{Y}_\infty$, together with the corresponding homological algebra, holds the promise of a satisfactory approach to the theory of "transversality conditions" in the most general case (see Section 8). In particular, it may be expected that we can find the generalization of the main principles of optimal control theory to the case of "multidimensional time" in this direction. Here, the interpretation of the Hamiltonian formalism in the calculus of variations in terms of this work, and, in particular, the interpretation along these lines of the work of Dedeker [5, 6], should be useful.

12.6. The standard equations of mechanics and mathematical physics undoubtedly deserve having their $\mathscr{C}$-spectral sequences described in sufficient detail. In particular, the computation of systematic tables of conservation laws and infinitesimal symmetries for these equations would open the way for a qualitative description of the phenomena modelled by them, in the way it is done in quantum physics by means of "quantum numbers."

12.7. One can expect to find effective methods for describing or estimating the term E_1 of the $\mathscr{C}$-spectral sequence by developing the techniques of Section 10. Methods for computing the terms E_r, $r > 1$ should be investigated within the framework of the category GDE by developing the standard methods of algebraic topology. In Section 10, for example, we showed the significance of homotopy techniques.

Note that in GDE a theory of fibre bundles, principal bundles, etc., be constructed. The development on this basis of the appropriate homology technique which, as is already clear now, is richer and more varied that the one used in similar situations in the theory of smooth manifolds, will undoubtedly lead to considerable progress in the problems under consideration.

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