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INCOMPLETE PAIRWISE COMPARISONS IN THE ANALYTIC HIERARCHY PROCESS

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Abstract—The Analytic Hierarchy Process is a decision-analysis tool which was developed by T. L. Saaty in the 1970s and which has been applied to many different decision problems in corporate, governmental and other institutional settings. The most successful applications have come about in group decision-making sessions, where the group structures the problem in a hierarchical framework and pairwise comparisons are elicited from the group for each level of the hierarchy. However, the number of pairwise comparison necessary in a real problem often becomes overwhelming. For example, with 9 alternatives and 5 criteria, the group must answer 190 questions. This paper explores various methods for reducing the complexity of the preference eliciting process. The theory of a method based upon the graph-theoretic structure of the pairwise comparison matrix and the gradient of the right Perron vector is developed, and simulations of a series of random matrices are used to illustrate the properties of this approach.

1. INTRODUCTION

The Analytic Hierarchy Process (AHP) was developed in the 1970s by T. L. Saaty [1] and over the years, has proven to be a very effective decision-analysis tool. Numerous application of this technique have included forecasting (inter- and intra-regional migration patterns, stock-market fluctuations etc.), investment decisions (portfolio selection, computer investment etc.) and socio-economic planning issues (transportation planning in Sudan, energy planning etc.). The essential ingredients in the AHP which lead to successful applications are the ability to incorporate "intangibles" into the decision-making process and its ease of use. In particular, applications of this technique to group decision making have proven to be most fruitful. In this type of situation, the group structures the problem in a hierarchical fashion, placing the overall objective of the decision at the top of the hierarchy and the criteria, subcriteria and decision alternatives on each descending level of the hierarchy. Once the group is satisfied with the problem structure, pairwise comparisons are elicited for each level of hierarchy in order to obtain the weights for each level with respect to one element in the next highest level in the hierarchy. For example, if the group is to choose one of four automobiles and three criteria are deemed to be important (style, handling, maintenance costs), then each criteria would be compared with all other criteria in a pairwise fashion with respect to the goal of purchasing the best car. Next, the automobiles would be compared according to each criteria. Finally, an overall weighting of the automobiles is obtained by synthesizing the weight from each level of the hierarchy; the book by Saaty [1] presents the theory of this process in detail.

The two advantages which the AHP has over other multi-criteria methods in this group setting are the ease of use and the ability to handle inconsistencies in judgments. People, acting unilaterally, are rarely consistent in their judgments. Thus, how can one ever expect a group to be consistent? The AHP does not force an individual or a group to be perfectly consistent when making pairwise comparisons, but incorporates the inconsistencies into the process decision maker a measure of the inconsistency in his/her/their judgments. The ability to handle inconsistency is a major point of the second advantage—the ease of use. Methods such as multi-attribute utility theory (MAUT) elicit transitive preferences at the cost of using complex eliciting mechanisms. The experience with the AHP supports Saaty's [1] claim that pairwise comparisons are somewhat "natural"; i.e. individuals or groups quickly become comfortable with the pairwise comparison mechanism and find it easy to use. By not forcing consistency of preferences, the AHP leads to a useful and usable decision-analysis tool.

The major drawback in the use of the AHP in either an individual or group decision process is the amount of work required to make all of the necessary pairwise comparisons. For example, if we have a problem of comparing 9 alternatives according to 5 criteria, a total of 190 pairwise comparison must be made. In realistic problems, this number is often quite higher. Thus, one comes to an important philosophical question concerning a decision-analysis tool: should the tool run the decision process or should the tool be considered to be a part of the process and not the process itself. It is the contention of this paper that, especially in group decision making, the latter must be the case. The structuring of the problem and the debate which precedes each pairwise comparison are vital aspects of the process which should not be curtailed due to time pressures arising from the need to complete all pairwise comparisons. Therefore, the purpose of this paper is to present a method for reducing the number of pairwise comparisons which must be made in an AHP session and thus, enable the group to focus on the debate and not the laborious task of completely filling in every comparison matrix.

There is another purpose for the development of a method to deal with incomplete pairwise comparisons. The AHP is based on the fact that pairwise comparisons are made on a ratio scale. Typically, the scale is bounded and the scale 1–9 is used, although any other scale could be used in this method [2]. Corresponding to this scale is a verbal description of the intensity of preference (equal, weak, moderate, strong, very strong, absolute). It is the intention of this research to lay the foundation for the development of a system with which the decision maker (a) only responds verbally and (b) is asked questions by the computer. Part (a) has already been implemented in systems such as *Expert Choice*. This paper presents the mathematical foundation for part (b); i.e. the development of an expert system-type implementation of the AHP. This system should be able to guide the decision maker in making the appropriate (i.e. important) judgments and to suggest that the decision maker stop making judgments after a certain number have already been made. Thus, a theory for dealing with incomplete pairwise comparisons must be developed in order to attain this expert system-like implementation of the AHP.

The remainder of this paper is structured as follows: Section 2 reviews the various methods which have been suggested for synthesizing a set of pairwise comparisons (least-squares, logarithmic least-squares and the eigenvector method), presents an argument for the use of the eigenvector method and discusses the problem of reducing the number of comparisons made in this method. Section 3 then presents the details of the method to deal with incomplete pairwise comparisons, Section 4 presents the results of a series of simulations using the proposed method and conclusions are drawn in Section 5.

2. SYNTHESIZING PAIRWISE COMPARISONS

Consider the problem of comparing a set of n alternatives with respect to a single criterion. Let $\mathbb{A} = (a_{ij})$ be the matrix of pairwise comparisons arising from this process, where

$$a_{ij} > 0 \quad \text{for } i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, n,$$

and

$$a_{ji} = 1/a_{ij} \quad \text{for } i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, n,$$

and $n = |\mathbb{A}|$. Thus, $\mathbb{A}$ is a positive reciprocal matrix of size n. Three methods have been suggested for synthesizing the set of pairwise comparisons to obtain a vector of attribute weights, $\mathbf{w} = (w_1, w_2, \ldots, w_n)^T$: least-squares (LS), logarithmic least-squares (LLS) and the eigenvector method (EM). The LS method [3] minimizes the Euclidean metric

$$\sum_{i=1}^{n} \sum_{j=1}^{n} (a_{ij} - w_i/w_j)^2 \tag{1}$$

to obtain the attribute weights $\mathbf{w}$, and the LLS method [4] minimizes

$$\sum_{i=1}^{n} \sum_{j=1}^{n} [\ln(a_{ij}) - \ln(w_i/w_j)]^2. \tag{2}$$

The EM, or Saaty's method, sets the attribute weights equal to the right principal eigenvector or right Perron vector of the matrix $\mathbb{A}$:

$$\mathbb{A}\mathbf{w} = \lambda_{\max}\mathbf{w}, \tag{3}$$

where $\lambda_{\max}$ is the principal eigenvector or Perron root of $\mathbb{A}$.

Which method should be used? Saaty and Vargas [5] have recently shown that the EM is the only one of the three methods which has desirable rank preservation properties. A positive reciprocal matrix $\mathbb{A} = (a_{ij})$ is said to be consistent if $a_{ij}\,a_{jk} = a_{ik}\ \forall\, i,j,k$. That is, $\mathbb{A}$ is consistent if all paths between any two vertices in the fully connected, directed graph corresponding to $\mathbb{A}$ have the same intensity, where the intensity of a path is defined to be the multiplication of all the arc intensities:

$$a_{ij} = a_{i,i1}a_{i1,i2}\cdots a_{ir,j}.$$

In the case of a consistent matrix, Saaty and Vargas [5] show that all methods yield the same attribute weights $\mathbf{w}$. This result is intuitive in that if $\mathbb{A}$ is consistent, only the $(n-1)$ judgments making up the top row of $\mathbb{A}$ (note $a_{ii} = 1\ \forall\, i$) are needed since all other matrix entries can be derived from the relation $a_{ij} = a_{i1}a_{1j}$. It is when the judgments are inconsistent that the methods diverge. In this case, the top triangular portion of the matrix, $n(n-1)/2$ judgments, must be completed. In this case, the EM is the only method which fully captures the rank ordering inherent in the data [5]. This result can also be intuited by the graph-theoretic interpretation of the EM. Saaty [1] has shown that the right Perron vector is just the average of the intensities of all paths starting at a particular alternative; i.e. the eigenvector is just the average dominance of an alternative over the other $n-1$ alternatives (see the example in Ref. [2] for a more complete presentation of this interpretation). Thus, the EM is an averaging process. The LS and LLS methods, on the other hand, use the implicit assumption that a decision maker minimizes some Euclidean measure of his inconsistency, equation (1) or (2), when choosing attribute weights. It is the author's belief that an averaging process is much more "natural" than imposing some metric and optimizing behavior on the problem. It is this EM which is at the heart of the AHP and which will be the subject of this paper.

If a decision maker were perfectly consistent, then only $(n-1)$ judgments must be elicited and $\lambda_{\max} \doteq n$ [1]. However, any inconsistencies would necessitate the completion of the top triangular portion of the matrix—$n(n-1)/2$ judgments. In this case, $\lambda_{\max} > n$ [1]. Thus, the index

$$\text{C.I.} = (\lambda_{\max} - n)/n \tag{4}$$

has been suggested by Saaty [1] as a measure of the inconsistency of the judgments. Typically, if C.I. $\leqslant 0.1$ the judgments are taken as acceptable and if C.I. > 0.1, the decision maker is urged to reconsider his or her judgments (see Ref. [6] for a description of this process and the suggestion of a new method by which the reconsideration of judgments takes place).

Since it is unknown whether or not a decision maker will be consistent, all $n(n-1)/2$ judgments must be elicited. Thus, the eigenvector method includes a great deal of redundancy in the sense that $n(n-1)/2$ judgments are elicited instead of the minimum number $(n-1)$. This redundancy plays a useful role in that a decision maker can incorrectly answer one pairwise comparison, but the final attribute weights will not be greatly affected due to the redundancy and the averaging effect of the EM. Therefore, one would not want to make only $(n-1)$ pairwise comparison since a certain amount of redundancy is necessary to "correct" any errors in the judgments. However, the completion of all $n(n-1)/2$ judgments is a laborious task. It is the purpose of this paper to explore ways by which a decision maker has some redundancy in his judgments, but does not need to make the complete set of pairwise comparisons. Hence, the remainder of this paper is devoted to the concept of *incomplete* pairwise comparison in the eigenvector and thus the AHP method.

As a footnote to this section, the only other authors to attempt to deal with the problem of using the AHP with a large number of alternatives are Weiss and Rao [7]. Their approach is essentially a factorial design of the comparisons. The approach which is detailed in this paper is to create a

"real-time" or expert system-like method and thus, is fundamentally different from the Weiss and Rao methodology.

3. INCOMPLETE PAIRWISE COMPARISONS

It is obvious that if all the pairwise comparisons are not made, then the LS and LLS methods can be easily generalized to this situation by restricting the indices on the summations in equations (1) and (2), respectively. One must only be sure to have at least one nonzero entry in each row of the pairwise comparison matrix $\mathbb{A} = (a_{ij})$; i.e. one must be sure to create at least a spanning tree in the directed graph $D(\mathbb{A})$ associated with the matrix $\mathbb{A}$. Thus, the graph $D(\mathbb{A})$ is no longer fully connected as is the case when all $n(n-1)/2$ comparisons are made, but it must at least be *connected* (see, for example, Ref. [8] for the definition of these graph-theoretic concepts). Given comparisons of this type, both the LS and LLS methods can be used. However, it was argued in the previous section that these methods are inferior to the EM and thus, a generalization to the EM to deal with incomplete comparisons is necessary.

Given a set of pairwise comparisons, not necessarily complete, which constitute a reciprocal matrix $\mathbb{A} = (a_{ij})$, the directed graph corresponding to the positive elements in $\mathbb{A}$ is a reflexive graph. Furthermore, it will be assumed that this graph is always connected. In this situation, what is the natural way to derive the attribute weights $\mathbf{w}$? Consider a matrix element $a_{ij} = 0$; i.e. a pairwise comparison which has not yet been made. For a reflexive connected graph there must exist at least one path from i to j. Thus, a natural way to fill in the missing matrix element would be to take the average of the intensities of all the possible elementary paths connecting i and j. That is, the judgment a_{ij} is the average of all the possible ways in which i and j can be judged by considering their relationship with intermediate attributes or nodes. If the incomplete judgments in $\mathbb{A}$ were perfectly consistent, then every elementary path from i to j must have the same intensity. With the presence of inconsistencies the intensity of each path may differ. In this case, an average of these path intensities must be taken. This average is *not* the arithmetic mean however. Aczél and Saaty [9] have proven that to synthesize group judgments, the *geometric mean* must be used in order to preserve the reciprocal property—if the synthesis of the judgments yield $a_{ij} = \alpha$, then the synthesis of the reciprocal of the judgments should yield $a_{ji} = 1/\alpha$. Since one can treat each path intensity as a separate judgment in a set of group judgments, the geometric mean of the path intensities must be used to synthesize this information to yield a_{ij}. Therefore, given a set of incomplete comparison which form a connected graph $D(\mathbb{A})$, the missing matrix elements in the top triangular portion of $\mathbb{A}$ are found by taking the geometric mean of the intensities of all the elementary paths connecting the two attributes in $D(\mathbb{A})$. The lower triangular position of this matrix is then calculated by the reciprocal property $a_{ji} = 1/a_{ij}$. Given the updated matrix $\mathbb{A}$, the weights can then be derived by the standard EM.

Say that one starts the process by eliciting n pairwise comparisons in such a way that the graph $D(\mathbb{A})$ is connected. Thus, some redundancy is included by asking one more question than the minimum, $n-1$. By following the procedure outlined above, a vector of attribute weights $\mathbf{w}$ can be derived. One could of course stop the process at this point and consider $\mathbf{w}$ to be the final vector of weights. However, it may be the case that either the decision maker is unhappy or uncomfortable with the current ranking in $\mathbf{w}$ or that the decision maker was highly inconsistent in the current set of comparisons. In either case, more comparisons need to be elicited. Thus, the question arises as to how to select the next comparison to be made. Of course, the decision maker may know which judgment is best in the sense that he or she is most confident in its value, in which case this comparison should be made next. However, it is more often the case that the decision maker must be guided to the next comparison. It is intuitively obvious in this case that the next question should be the one which has the greatest impact on the weighting $\mathbf{w}$; i.e. the next question should be the one which in some way is related to the largest absolute gradient of $\mathbf{w}$ with respect to the unknown matrix elements. The choice of such a question will be detailed in a moment but first, formulas for the gradient of $\mathbf{w}$ with respect to a matrix element a_{ij} must be derived.

Consider the class of positive reciprocal (square) matrices $\mathbb{A} = (a_{ij})$

$$\Lambda^{n,n} = \{\mathbb{A} = (a_{ij}) \in R^{n,n} | a_{ij} > 0 \quad \forall\, 1 \leqslant i,j \leqslant n; a_{ji} = 1/a_{ij} \quad \forall\, 1 \leqslant i,j \leqslant n\},$$

and consider the following eigenvector problem:

$$\mathbb{A}\mathbf{x}(\mathbb{A}) = r(\mathbb{A})\mathbf{x}(\mathbb{A}) \tag{5}$$

where $r(\mathbb{A}) = \lambda_{\max}$ is the Perron root or principal eigenvalue of $\mathbb{A}$ and $\mathbf{x}(\mathbb{A})$ is the right Perron vector of $\mathbb{A}$. The author [6] has recently proven the following results on the derivatives of $r(\mathbb{A})$ with respect to a matrix element in the upper triangular portion of $\mathbb{A}$.

Lemma 1

Let $\mathbb{A} \in \Lambda^{n,n}$. Then for $j > i$, $\partial\mathbb{A}/\partial_{ij}$ is an $n \times n$ matrix of the form

$$\left[\frac{\partial\mathbb{A}}{\partial_{ij}}\right]_{kl} = \begin{cases} 1 & \text{if } k = i, l = j \\ -1/(a_{ij})^2 & \text{if } k = j, l = i \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2

Let $\mathbb{A} \in \Lambda^{n,n}$. Then $\mathbb{D}_r^{\mathbb{A}}$ is an $n \times n$ upper triangular matrix of the form

$$\begin{aligned} \mathbb{D}_r^{\mathbb{A}} &= \left[\frac{\partial r(\mathbb{A})}{\partial_{ij}}\Big|_{j>i}\right] \\ &= [[\mathbf{y}(\mathbb{A})_i\mathbf{x}(\mathbb{A})_j] - [\mathbf{y}(\mathbb{A})_j\mathbf{x}(\mathbb{A})_i]/[a_{ij}]^2\big|_{j>i}], \end{aligned} \tag{6}$$

where $\mathbf{x}(\mathbb{A})$ and $\mathbf{y}(\mathbb{A})$ are, respectively, the right and left Perron vectors of $\mathbb{A}$ and $\mathbf{y}(\mathbb{A})^{\mathrm{T}}\mathbf{x}(\mathbb{A}) = 1$.

Using these results, the following theorem on the derivatives of $\mathbf{x}(\mathbb{A})$ with respect to a matrix element in the upper triangular portion of $\mathbb{A}$ can be proven.

Theorem

Let $\mathbb{A} \in \Lambda^{n,n}$ and let $r(\mathbb{A})$, $\mathbf{x}(\mathbb{A})$, $\mathbf{y}(\mathbb{A})$ denote respectively, the Perron root and right and left vectors of $\mathbb{A}$. Then

$$\begin{aligned} \mathbb{D}_{\mathbf{x}}^{\mathbb{A}} &= \left[\frac{\partial \mathbf{x}(\mathbb{A})}{\partial_{ij}}\Big|_{j>i}\right] \\ &= \left[\begin{bmatrix} \tilde{\mathbb{A}} - r(\mathbb{A})\tilde{\mathbb{I}} \\ \mathbf{e} \end{bmatrix}^{-1} \begin{bmatrix} \tilde{\mathbb{D}}_r^{\mathbb{A}}\mathbf{x}(\mathbb{A}) - \tilde{\mathbf{Z}}(\mathbb{A}) \\ 0 \end{bmatrix}\right], \end{aligned} \tag{7}$$

where $\mathbb{I}$ is the $n \times n$ identity matrix, $\mathbf{e}$ is an n-dimensional row vector of ones, $\mathbf{Z}(\mathbb{A}) = (\mathbf{Z}_k)$ is an n-dimensional column vector defined as follows:

$$\mathbf{Z}_k = \begin{cases} \mathbf{x}_j(\mathbb{A}) & \text{if } k = i \\ -\mathbf{x}_i(\mathbb{A})/(a_{ij})^2 & \text{if } k = j \\ 0 & \text{otherwise;} \end{cases}$$

and "$\sim$" denotes the matrix or vector with its last row deleted.

Proof. It is well-known that the Perron root of a matrix $\mathbb{A} \in \Lambda^{n,n}$ is simple and thus, there exists a neighborhood $N_{\mathbb{A}}$ of $\mathbb{A}$ in $R^{n,n}$ such that each $\mathbb{B} \in N_{\mathbb{A}}$ has a simple eigenvalue $\lambda(\mathbb{B})$ and such that for $\mathbb{B} \in N_{\mathbb{A}} \cap \Lambda^{n,n}$ we have $\lambda(\mathbb{B}) = r(\mathbb{B})$ [6]. Furthermore, $\lambda(\cdot)$ is analytic as a function of the n^2 entries of the elements in $N_{\mathbb{A}}$ and thus, the partial derivatives of all orders of $\lambda(\cdot)$ with respect to the n^2 matrix elements must exist and be well-defined. For each $\mathbb{B} \in N_{\mathbb{A}}$, let $\mathbf{x}(\mathbb{B})$ be the right eigenvector of $\mathbb{B}$ corresponding to $\lambda(\mathbb{B})$:

$$\mathbb{B}\mathbf{x}(\mathbb{B}) = \lambda(\mathbb{B})\mathbf{x}(\mathbb{B}), \tag{8}$$

for which

$$\mathbf{e}\mathbf{x}(\mathbb{B}) = 1, \tag{9}$$

where $\mathbf{e}$ is a row vector of ones. Thus, $\mathbf{x}(\cdot)$ is analytic as a function of each of the elements of $N_{\mathbb{A}}$, and the partial derivatives $\partial\mathbf{x}(\mathbb{B})/\partial_{ij}$ of $\mathbf{x}(\cdot)$ at $\mathbb{B}$ with respect to the matrix elements must exist and be well-defined. Now let $\mathbb{B} = (b_{ij}) \in N_{\mathbb{A}}$ and consider the following:

$$\mathbb{B}\mathbf{x}(\mathbb{B}) = r(\mathbb{B})\mathbf{x}(\mathbb{B}). \tag{10}$$

Differentiating equation (10) on both sides by the (i,j)th entry $1 \leqslant i, j \leqslant n, j > i$, one obtains

$$\frac{\partial\mathbb{B}}{\partial_{ij}}\mathbf{x}(\mathbb{B}) + \mathbb{B}\frac{\partial\mathbf{x}(\mathbb{B})}{\partial_{ij}} = \frac{\partial r(\mathbb{B})}{\partial_{ij}}\mathbf{x}(\mathbb{B}) + r(\mathbb{B})\frac{\partial\mathbf{x}(\mathbb{B})}{\partial_{ij}}. \tag{11}$$

Using the results of Lemmas 1 and 2, this equation can be rewritten as

$$\begin{aligned}[\mathbb{B} - r(\mathbb{B})\mathbb{I}]\frac{\partial\mathbf{x}(\mathbb{B})}{\partial_{ij}} &= \frac{\partial r(\mathbb{B})}{\partial_{ij}}\mathbf{x}(\mathbb{B}) - \frac{\partial\mathbb{B}}{\partial_{ij}}\mathbf{x}(\mathbb{B}) \\ &= \mathbb{D}_r^{\mathbb{B}}\mathbf{x}(\mathbb{B}) - \mathbf{Z}(\mathbb{B}),\end{aligned} \tag{12}$$

where $\mathbf{Z}(\mathbb{B}) = (\mathbf{Z}_k)$ is an n-dimensional column vector of the form

$$\mathbf{Z}_k = \begin{cases} \mathbf{x}_j(\mathbb{B}) & \text{if } k = i \\ -\mathbf{x}_i(\mathbb{B})/(b_{ij})^2 & \text{if } k = j \\ 0 & \text{otherwise.} \end{cases}$$

By definition, the matrix $[\mathbb{B} - r(\mathbb{B})\mathbb{I}]$ is singular. However, since

$$\mathbf{e}\mathbf{x}(\mathbb{B}) = 1,$$

it follows that

$$\mathbf{e}\frac{\partial\mathbf{x}(\mathbb{B})}{\partial_{ij}} = 0. \tag{13}$$

Deleting the last row in $[\mathbb{B} - r(\mathbb{B})\mathbb{I}]$, $\mathbb{D}_r^{\mathbb{B}}$ and $\mathbf{Z}(\mathbb{B})$ and adding the row vector $\mathbf{e}$ to the l.h.s. of equation (12) and 0 to the r.h.s. yields

$$\begin{bmatrix} \tilde{\mathbb{B}} - r(\mathbb{B})\tilde{\mathbb{I}} \\ \mathbf{e} \end{bmatrix}\frac{\partial\mathbf{x}(\mathbb{B})}{\partial_{ij}} = \begin{bmatrix} \tilde{\mathbb{D}}_r^{\mathbb{B}}\mathbf{x}(\mathbb{B}) - \tilde{\mathbf{Z}}(\mathbb{B}) \\ 0 \end{bmatrix}, \tag{14}$$

where "$\tilde{\ }$" denotes the same vector or matrix with its last row deleted. The matrix on the l.h.s. equation (14) will now be nonsingular and, by letting $\mathbb{B} = \mathbb{A}$, the conclusion of the theorem is obtained.† Q.E.D.

The above theorem gives one a means by which the gradients of the right Perron vector and hence the attribute weights $\mathbf{w}$ can be easily calculated from the right and left Perron vectors. How does one now use this information to guide the decision maker to the next comparison, and how

† This formula is an extension of the type of results obtained by Wilkinson [10] and Vargas [11] on perturbation to essentially nonnegative and positive reciprocal matrices.

is this information used to devise stopping rules i.e. rules for terminating the pairwise comparisons before all $n(n-1)/2$ comparisons are made? These two questions will now be addressed.

The logical choice of the next question would be the cell entry which has the greatest impact on the attribute weights; i.e. the comparison with the largest absolute gradient of the right Perron vector. Obviously, one would not want to ask a question which had little influence on the weights. Thus, the choice of the next comparison (i,j) by the rule

$$(i,j) = \underset{(k,l)\in Q}{\operatorname{argmax}} (\|\partial \mathbf{x}(A)/\partial_{kl}\|_{\infty}), \tag{15}$$

where Q is the set of unanswered comparisons and $\|\cdot\|_{\infty}$ denotes the L_{∞} or Tchebyshev norm, will direct one to the most important question. Of course, one should not force the decision maker to choose this comparison and not consider any others, but one should present the decision maker with a ranking of the unanswered comparison in terms of equation (15) and allow him to select the next comparison. This ranking, however, is vital for the decision maker since it gives him information on the importance of the remaining comparisons.

The next issue involves the decision to stop making pairwise comparisons. There are three possible ways which this decision can be made. The first is to let the decision maker decide whether or not to continue with the questioning. In fact, this option is always available under the other two stopping rules. The second rule would state that if the maximum absolute difference in the attribute weights from one question to the next is $\leqslant \alpha\%$, where α is a given constant, then one should stop since the new comparison did not have a major influence on your weighting. Formally, if $\mathbf{w}^k$ and $\mathbf{w}^{k+1}$ are, respectively, the attribute weights after k and $k+1$ comparisons have been made and

$$l = \underset{1\leqslant i\leqslant n}{\operatorname{argmax}} |w_i^{k+1} - w_i^k|/w_i^k, \tag{16}$$

then the procedure would stop at $k+1$ comparisons if

$$\frac{|w_l^{k+1} - w_l^k|}{w_l^k} \leqslant \alpha. \tag{17}$$

This rule is very "liberal" in the sense that further questioning may drastically alter the weights. However, the decision maker always has a veto power and hence this rule may work well in practice.

The third stopping rule is very conservative in the sense that comparisons will continue to be made until one is sure that ordinal rank will not be reversed. The weights $\mathbf{w}$ are cardinal rankings of the alternatives which, of course, create an ordinal ranking of the alternatives. By answering more questions, the cardinal ranking in $\mathbf{w}$ may be slightly altered but the ordinal ranking could remain the same. The third stopping criterion states that the next question derived from the gradient procedure just described will only be asked if it appears that the ordinal ranking could be reversed. More precisely, consider a current value a_{ij} of the (i,j)th question which has just been chosen as the next question to be asked. Let $U_{ij} = \max(1, \bar{a}_{ij} - a_{ij})$, where $\bar{a}_{ij}$ is the largest path intensity in the set of all elementary paths connecting i and j, and let $L_{ij} = \max(1, a_{ij} - \underline{a}_{ij})$, where $\underline{a}_{ij}$ is the smallest path intensity. If the decision maker was perfectly consistent up to the current comparison, then $\bar{a}_{ij} = a_{ij} = \underline{a}_{ij}$. However, one cannot be sure that the decision maker will not be at least slightly inconsistent in the next question and thus, a perturbation of 1 is introduced. For example, if the current value of $a_{ij} = 6$ and $\bar{a}_{ij} = 9, \underline{a}_{ij} = 5.4$, then $U_{ij} = 3$ and $L_{ij} = 1$. One cannot assume that perfect consistency in a subset of comparisons is a valid criterion for stopping the process since there is always the possibility that perturbations could occur. By choosing L_{ij} and U_{ij} in the way which is described above, one allows for these perturbations. Given these upper and lower bounds on the possible deviation from a_{ij}, let us define $P(\mathbf{w})$ to be a function which returns the ordinal ranking inherent in the cardinal ranking $\mathbf{w}$; i.e. $P: R^n \to Z^n$ where Z^n is the n-dimensional space of

natural numbers. For example, if $\mathbf{w} = (0.15,\ 0.3,\ 0.2,\ 0.35)^T$ then $P(\mathbf{w}) = (4, 2, 3, 1)^T$. Using this function, three ranking can be defined:

$$P_1 = P(\mathbf{w}),$$

$$P_2 = P\left(w + \frac{\partial w}{\partial_{ij}}(a_{ij} + U_{ij})\right)$$

and

$$P_3 = P\left(w + \frac{\partial w}{\partial_{ij}}(a_{ij} - L_{ij})\right).$$

Ranking P_1 is the current ordinal ranking, and rankings P_2 and P_3 are the approximations to the ordinal rankings which would occur if the (i,j)th comparison achieved its maximum and minimum deviation, respectively. If $P_1 = P_2 = P_3$, then it is likely that the next comparison will not alter the ordinal ranking inherent in $\mathbf{w}$ and hence, the procedure may be terminated. This ordinal rank reversal criterion is very conservative in the sense that two alternatives may have low but almost equivalent weights and this criterion would not terminate the comparisons in these circumstances. Alternatives with low weights are not important and thus one would like to ignore a possible rank reversal in this situation. However, the criterion described above would force the eliciting process to continue. Therefore, one must either consider using this stopping criterion, a stopping criterion such as the above mentioned $\alpha\%$ rule and making it possible for the decision maker to decide to stop, or continue this process, or some combination of these three rules; which rule is best becomes a purely empirical question which will be explored in future research.

In the actual implementation of the procedure outlined above, computational considerations call for a modification to this method. For an incomplete comparison method to be useful to a decision maker, the computation of the eigenvector and derivatives after each question must be done quickly. The most computationally burdensome task in this step is the computation of all the elementary paths between two specified vertices in $D(\mathbb{A})$. As Carré [8] points out, this problem can be solved via a backtracking algorithm. However, as the number of completed comparison grows, the number of elementary paths grows exponentially. Thus, the determination of all elementary paths becomes extremely difficult. Due to the computational complexity of this task, a simplification will be made. Instead of finding all elementary paths, a sample of random spanning trees will be used to calculate a_{ij}, $\bar{a}_{ij}$ and $\underline{a}_{ij}$. Finding the shortest spanning tree in a graph is extremely easy, so a procedure in which arc costs are randomly derived will be used in conjunction with a shortest spanning tree algorithm to derive a sample of elementary paths.

In summary, the steps used in the incomplete pairwise comparison method are:

Step 0. Have the decision maker provide n judgments which form a connected graph $D(\mathbb{A})$.

Step 1. Using the completed pairwise comparisons, derive the missing comparison by taking the geometric mean of intensities of a sample of random spanning trees. Calculate the weight $\mathbf{w}$.

Step 2. Calculate the derivatives of $\mathbf{w}$ with respect to the missing matrix elements and select the next question according to equation (15).

Step 3. If this question meets the appropriate stopping criteria (subjective assessment, $\alpha\%$ rule, ordinal rank rule etc.), stop; else elicit this comparison and return to Step 1.

As a final comment on this section, the results of the theorem can also be used in a sensitivity analysis at the end of the process. By being easy to compute, the derivatives of $\mathbf{w}$ with respect to the matrix elements can be quickly used to guide to decision maker in revising any judgments which were made during the course of this process.

Table 1

Distance from Philadelphia	1	2	3	4	5	6	w
1. Cairo	1	1/3	8	3	3	7	0.2619
2. Tokyo		1	9	3	3	9	0.3975
3. Chicago			1	1/6	1/5	2	0.0334
4. San Francisco				1	1/3	6	0.1164
5. London					1	6	0.1642
6. Montreal						1	0.0266

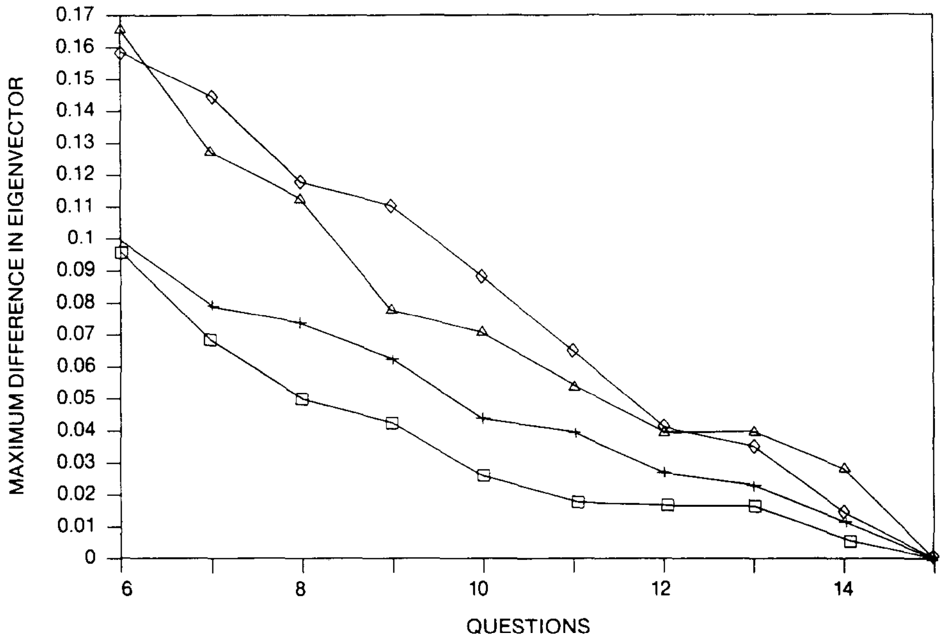

Fig. 1. Results of matrices of size $N = 6$.

4. NUMERICAL RESULTS

In order to obtain some insight as to the possible benefits which the proposed method might yield, a series of numerical experiments were performed. First, consider the example in Saaty [1] of using the eigenvector method to predict the relative distance of a set of cities from Philadelphia (see Table 1). Consider the situation where the first 6 comparisons are (1, 2), (1, 3), (2, 5), (3, 6), (4, 5) and (5, 6). The process described in Section 3 yields the results presented in Table 2 (there are 15 comparisons in total). The 5% stopping rule would say to stop at question 6 while the rank order criterion would stop the procedure at question 10. In either case, it is obvious from the table that not all 15 questions are necessary. Thus, the procedure outlined in this paper can yield significant time savings in this example.

In order to test this procedure more thoroughly, a simulation of 50 random matrices of size 6, 7, 8 and 9 was performed; Figs 1–4 show the results of this experiment. In these figures, the maximum difference in the eigenvector is defined as $\|\mathbf{w}^k - \mathbf{w}^*\|_\infty$, where $\mathbf{w}^k$ is the eigenvector after

Table 2

Question	w_1	w_2	w_3	w_4	w_5	w_6
6	0.2339	0.4612	0.0382	0.0659	0.1734	0.0273
7	0.2237	0.4649	0.0394	0.0688	0.1757	0.0275
8	0.2863	0.4504	0.0407	0.0474	0.1464	0.0288
9	0.2769	0.4435	0.0321	0.0785	0.1433	0.0257
10	0.2200	0.4004	0.0349	0.1361	0.1771	0.0315
11	0.2684	0.3855	0.0325	0.1254	0.1641	0.0240
12	0.2694	0.4003	0.0338	0.1071	0.1622	0.0273
13	0.2686	0.3991	0.0335	0.1104	0.1622	0.0262
14	0.2676	0.3970	0.0330	0.1149	0.1623	0.0251
15	0.2619	0.3975	0.0334	0.1164	0.1642	0.0266

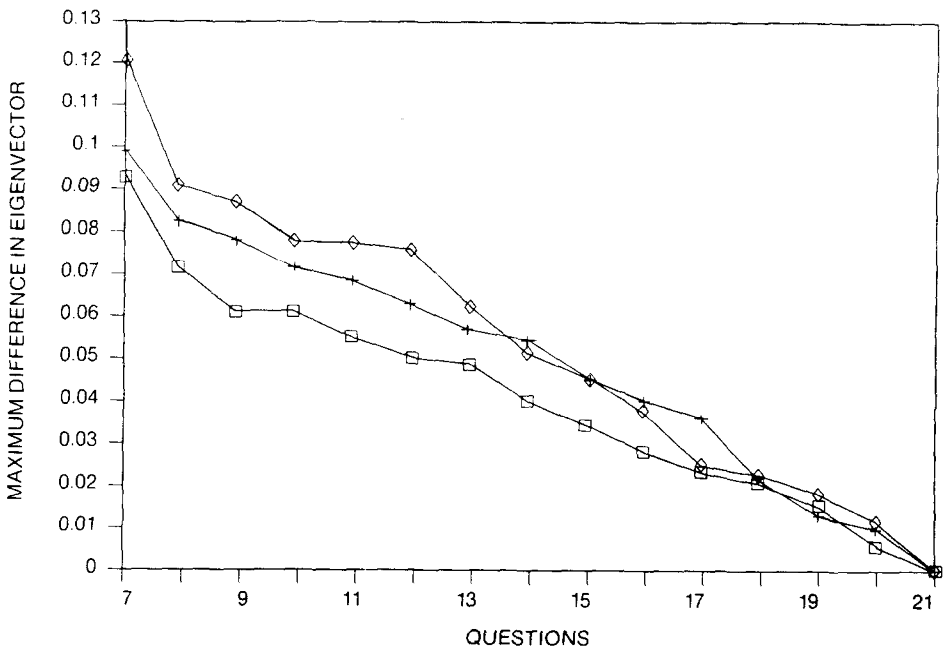

Fig. 2. Results of matrices of size $N = 7$.

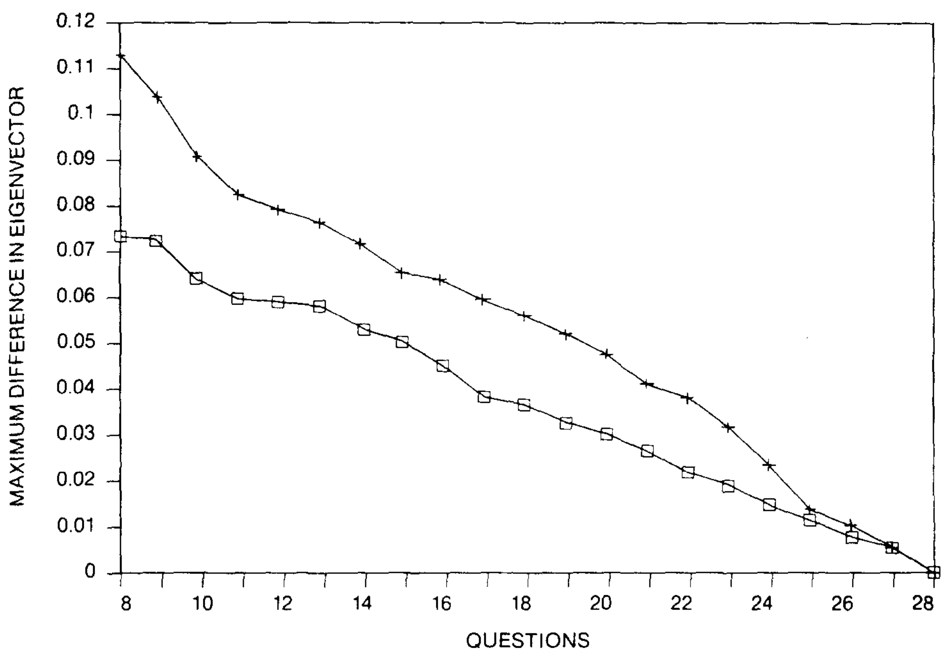

Fig. 3. Results of matrices of size $N = 8$.

question k has been answered and $\mathbf{w}^*$ is the eigenvector when the matrix is complete. As one can see, the errors increase with increasing inconsistency (C.I.) as expected. Also, the errors tend to fall more rapidly at the beginning of the process and tend to fall very slowly as k approaches $n(n - 1)/2$. This result is also expected since the process is first choosing those questions with the greatest impact on the eigenvector. Therefore, as k approaches $n(n - 1)/2$, the comparison tend to become less and less useful which confirms the belief that it is not worthwhile to make *all* the comparisons in the EM. Finally, the average number of questions which need to be asked under the 5% and ordinal rank stopping criteria for the various size matrices are as given in Table 3. One can immediately see how conservative the ordinal ranking rule is in practice and the liberality of the

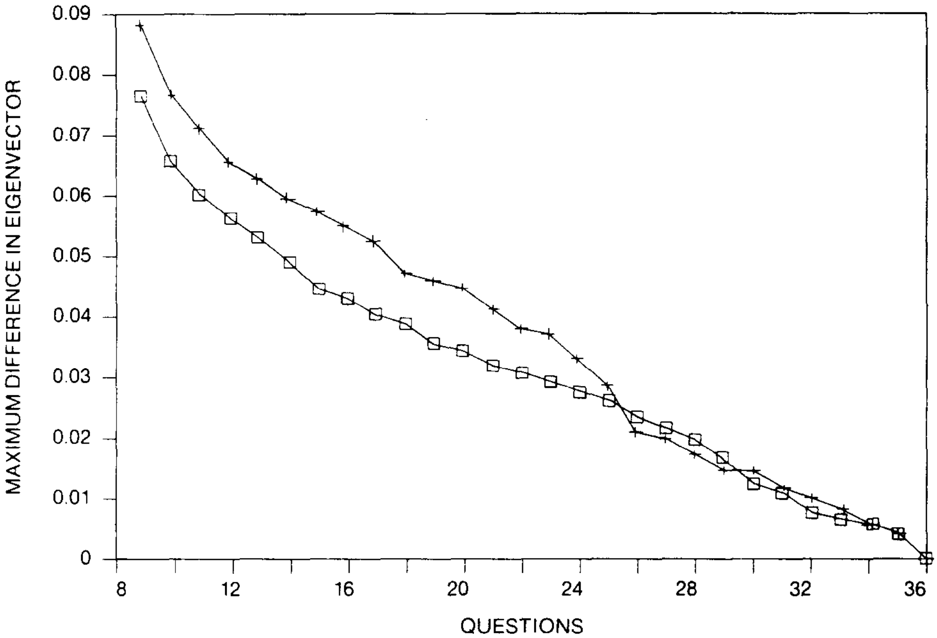

Fig. 4. Results of matrices of size $N = 9$.

Table 3

n	5% Rule	Percentage of total No. of questions	Ordinal ranking rule	Percentage of total No. of questions
6	10.90	72.67	12.60	84.00
7	12.24	58.29	17.18	81.81
8	13.56	48.43	22.78	81.36
9	14.36	39.89	30.68	85.22

α% rule. Also, there is a dramatic decrease in the percentage of questions which must be answered under the 5% rule. In practice, these results may not be as striking, but these results clearly indicate that the incomplete pairwise comparison process described in this paper can substantially reduce the work involved in the EM.

5. CONCLUSION

This paper has developed a method by which substantial time savings in using the AHP can be achieved. These time savings are important in that they simplify the work involved in making the pairwise comparisons and therefore, give the individual or group of decision makers more time to debate certain judgments and create different hierarchical structures for the problem which can then be compared and synthesized. Thus, the incomplete pairwise comparison method presented in this paper helps remove the decision-analysis tool as the primary focus of the decision process and puts it in being an aid to the process.

At least two research items emerge from this study. The first is to answer the question of how well this process will work in practice. A series of empirical experiments must be performed in order to validate and refine the proposed method. The second involves the hierarchical structure. The AHP currently asks the decision maker to make pairwise comparisons of one criteria at a time. The method described in this paper simply reduces the amount of work needed under each criteria. There is another way of looking at the process. Instead of comparing all alternatives under each criteria, one could compare two alternatives under all criteria. Both a theoretical question as to how the method proposed in this paper can deal with this reversal in the comparison process, especially when there are more than two levels in the hierarchy and an empirical question of the ability of a decision maker to congnitively process information in this new frame of reference remain to be answered.

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REFERENCES

1. T. L. Saaty, *The Analytic Hierarchy Process.* McGraw-Hill, New York (1980).
2. P. T. Harker and L. G. Vargas, The theory of ratio scale estimation: Saaty's Analytic Hierarchy Process. *Mgmt Sci.* (in press).
3. K. O. Cogger and P. L. Yu, Eigen weights vectors and least distance approximation for revealed preference in pairwise weight ratios. Unpublished paper, School of Business, Univ. of Kansas, Lawrence, Kan. (1983).
4. J. G. DeGraan, Extensions of the multiple criteria analysis method of T. L. Saaty. Paper presented at *EURO IV*, Cambridge, U.K. (1980).
5. T. L. Saaty and L. G. Vargas, Inconsistency and rank preservation. *J. math. Psychol.* **28**, 205–214 (1984).
6. P. T. Harker, Derivatives of the Perron root of a positive reciprocal matrix: with application to the Analytic Hierarchy Process. *Appl. Math. Computn* **22**, 217–232 (1987).
7. E. N. Weiss and V. R. Rao, AHP design issues for large scale systems. *Decis. Sci.* **18**, 43–61 (1987).
8. B. Carré, *Graphs and Networks.* Clarendon Press, Oxford (1979).
9. J. Aczél and T. L. Saaty, Procedures for synthesizing ratio judgments. *J. math. Psychol.* **27**, 93–102 (1983).
10. J. H. Wilkinson, *The Algebraic Eigenvalue Problem.* Oxford Univ. Press, London (1965).
11. L. G. Vargas, Analysis of sensitivity of reciprocal matrices. *Appl. Math. Computn* **12**, 301–302 (1983).