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A sheaf-theoretic foundation for nonstandard analysis

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Abstract

A new foundation for constructive nonstandard analysis is presented. It is based on an extension of a sheaf-theoretic model of nonstandard arithmetic due to I. Moerdijk. The model consists of representable sheaves over a site of filter bases. Nonstandard characterisations of various notions from analysis are obtained: modes of convergence, uniform continuity and differentiability, and some topological notions. We also obtain some additional results about the model. As in the classical case, the order type of the nonstandard natural numbers is a dense set of copies of the integers. Every standard set has a hyperfinite enumeration of its standard elements in the model. All arguments are carried out within a constructive and predicative metatheory: Martin-Löf's type theory.

1. Introduction

We present a new approach to constructive nonstandard analysis, which is based on an expansion of Moerdijk's sheaf-theoretic nonstandard model of arithmetic [6]. Using the model we improve on the results of our earlier constructive approach [8, 9]. In particular, many important nonstandard characterisations of analytic notions can be regained from classical nonstandard analysis. One of the main points of the present paper is that the results are established within a completely constructive and predicative framework. We may thus use the model to extend the methods of Bishop's constructive analysis [1].

Recall that a classical construction of nonstandard real numbers may be obtained by taking a reduced power of the ordinary real numbers modulo a fixed nonprincipal ultra filter. A reduced power model was also used in our previous approach, but the power was then taken modulo the (constructive) Fréchet filter. The model of Moerdijk may

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loosely speaking be described as a reduced power with a variable filter structure. A crucial property of this model is that the *full transfer principle* holds. The principle depends on a Łos-type theorem. To prove this theorem without using (nonconstructive) ultra filters, there is need to compare, split and extend filters in a flexible manner, as one works ones way through the clauses of sheaf semantics. This is the reason for introducing a whole category of filter bases in Section 2. The nonstandard model consists, more precisely, of representable sheaves over a site given by this category. Every small mathematical object (in the sense of category theory) has a nonstandard counterpart in the model. The model is studied in Section 3. We prove that the order type of the nonstandard natural numbers is a dense set of copies of the integers, and that the standard elements of each standard set has a hyperfinite enumeration in the model. Moreover, some negative results about the existence of standard parts of real numbers are obtained. Then in Section 4, nonstandard characterisations of various modes of convergence, uniform continuity and differentiability and topological notions, including some examples of their use, are finally provided.

1.1. Preliminaries

The framework of our development will be Martin-Löf's type theory with a universe of types. This theory is fully adequate to formalise constructive analysis. Since Martin-Löf's type theory has a straightforward interpretation into set theory (with one extra universe of sets), the results of the present paper may easily be understood classically as well.

Let (U, T) denote the universe of types, where U is the set of codes (or small type symbols) and T is the decoding function. A type S, or *set* as we prefer to say, is *small* if it has a code in the universe, i.e. $S \equiv T(s)$ for some $s \in U$. A relation R on a small set S, is *small* if $R(s) \equiv T(r(s))$ for some $r \in S \to U$. Similarly, a function is *small* if its domain and codomain are small. Every set A of type theory (small or not) comes equipped with a *basic* equality relation $Id(A, \cdot, \cdot)$, which we do not assume to be extensional as in [4]. However, we need to consider coarser equivalence relations on sets. We use $=$, possibly with a subscript, to denote such construed equalities, and reserve the symbol $\equiv$ for definitional equality. The mathematical structures live in the category of sets with (construed) equality, where the morphisms or functions between such sets are required to respect the equalities. A set with equality is *finite* if it is isomorphic to some canonical finite set $N_k \equiv \{0, \dots, k-1\}$ which is equipped with the basic equality relation. It will be convenient to use set-theoretic language when dealing with predicates on a fixed set. When P and Q are predicates on S, we use the dot notation $P \dot{\subseteq} Q$ for $(\forall s \in S)[P(s) \to Q(s)]$. In the same manner, set operations $\cap$, $\cup$ and predicative set former $\{x \in S : \cdots x \cdots\}$ are used to form new predicates. We take $s \dot{\in} P$ as alternate notation for $P(s)$, and often abbreviate $(\forall s \in S)[P(s) \to \cdots]$ by $(\forall s \dot{\in} P)[\cdots]$ when S can be inferred from the context. A similar convention is adopted for existential quantification.

In this paper we will assume familiarity with the basics of sheaf semantics and forcing over sites (cf. [3]). A detailed exposition of sheaf semantics from a constructive and predicative point of view is made in [10].

2. The category of filter bases

We define the *category of filter bases* as follows. A *filter base* $\mathscr{F} \equiv (I, =_I, J, =_J, F)$ consists of an inhabited *index set* I and an arbitrary set J with equivalence relations $=_I$ and $=_J$, respectively, and a two place relation

$$F(i,j) \qquad (i \in I, j \in J)$$

which respects these equivalences, and further satisfies the filter condition:

$$(\forall i_1, i_2 \in I)(\exists k \in I) F_k \mathrel{\dot{\subseteq}} F_{i_1} \cap F_{i_2},$$

where we write $j \mathrel{\dot{\in}} F_i$ for $F(i,j)$. One says also that F is a *filter base on* $(J, =_J)$.

Example 2.1. Letting $F(i,j) \equiv j \geqslant i$, with $I = J = \mathbb{N}$, yields a filter base for the Fréchet filter.

A *map* between filter bases $\mathscr{F} \equiv (I, =_I, J, =_J, F)$ and $\mathscr{F}' \equiv (I', =_{I'}, J', =_{J'}, F')$ is a binary relation

$$\alpha(j, j') \qquad (j \in J, j' \in J')$$

respecting $=_J$ and $=_{J'}$ in its arguments and which is total and functional on some base set F_i, i.e. more precisely for some $i \in I$,

$$(\forall j \mathrel{\dot{\in}} F_i)(\exists j' \in J')[\alpha(j,j') \wedge (\forall j'' \in J')(\alpha(j,j'') \rightarrow j' =_{J'} j'')].$$

When it is clear from the context that α can be assumed to be functional we will, in later sections, use the notation $\alpha(j')$ for the j' such that $\alpha(j,j')$. The map is *continuous* if for each $k \in I'$ there is some $i \in I$ with

$$(\forall x \mathrel{\dot{\in}} F_i)(\exists y \mathrel{\dot{\in}} F'_k)\, \alpha(x,y),$$

or more briefly $\alpha[F_i] \mathrel{\dot{\subseteq}} F'_k$. The continuous maps are the morphisms of the category. Two maps $\alpha, \beta : \mathscr{F} \to \mathscr{F}'$ are said to be *equal* (in symbols $\alpha \sim \beta$) if for some $i \in I$,

$$(\forall j \mathrel{\dot{\in}} F_i)(\forall \ell \in J')[\alpha(j,\ell) \leftrightarrow \beta(j,\ell)].$$

This is an equivalence relation on maps, by the filter condition. For any filter base $\mathscr{F}$ we define a continuous map corresponding to the identity, $\delta_{\mathscr{F}}(j,k) \equiv (j =_J k)$. The composition $\circ$ of two morphisms is defined to be the obvious relational composition.

It is straightforward to check that this indeed defines a category. There is an obvious small version of this category called $\mathbb{B}$ where all the sets and relations are small.

Lemma 2.2. *The category $\mathbb{B}$ has terminal object and all pullbacks.*

Proof. As terminal object we can take $(I,=_I,J,=_J,F)$ where $I \equiv J \equiv N_1$ is the canonical one element set, and the relation $F(i,j)$ is always true, and $(a =_I b) \equiv (a =_J b) \equiv Id(N_1,a,b)$, the identity relation on N_1.

Given two morphisms $\alpha' : \mathscr{F}' \to \mathscr{F}$ and $\alpha'' : \mathscr{F}'' \to \mathscr{F}$, define the pullback object $\mathscr{G} \equiv (I' \times I'', =_{I'\times I''}, J' \times J'', =_{J'\times J''}, G)$ by

$$G((i_1,i_2),(x,y)) \equiv F'(i_1,x) \wedge F''(i_2,y) \wedge (\exists z \in I)\, \alpha'(x,z) \wedge \alpha''(y,z),$$

and the equality $=_{I'\times I''}$ is taken to be the componentwise equality obtained from $=_{I'}$ and $=_{I''}$, and similarly for $=_{J'\times J''}$. $\square$

We now define a site on the category $\mathbb{B}$ by giving its basis of covering families. Let $\langle \varphi_k : \mathscr{F}^{(k)} \to \mathscr{F} \rangle_{k\in S}$ be a finite family of morphisms in $\mathbb{B}$, where $\mathscr{F}^{(k)} \equiv (I^{(k)}, =_{I^{(k)}}, J^{(k)}, =_{J^{(k)}}, F^{(k)})$. The family is a *cover* if for any choice of indices $i_k \in I^{(k)}$ $(k \in S)$ there exists an index $j \in I$ such that

$$(\forall y \mathrel{\dot\in} F_j)(\exists k \in S)(\exists z \in J^{(k)})\, \varphi_k(z,y) \wedge z \mathrel{\dot\in} F^{(k)}_{i_k},$$

or expressed more briefly, $F_j \mathrel{\dot\subseteq} \bigcup_{k\in S} \varphi_k[F^{(k)}_{i_k}]$. The cover relation is denoted by K.

The proofs of the following results are analogous to those found in [7, Theorem 4.3].

Lemma 2.3. *The cover relation K is a basis for a site on $\mathbb{B}$.*

Lemma 2.4. *Each representable presheaf over $(\mathbb{B},K)$ is a sheaf.*

2.1. Nonstandard objects

The nonstandard counterparts to the small sets, functions and constants of type theory will now be defined. Any small set S with equality $=_S$ gives rise to a *simple* or *trivial* filter base, $\hat{S} = (N_1, =_{N_1}, S, =_S, F)$, where $F(j,s)$ is always true. In view of Lemma 2.4, we obtain from this a *simple sheaf* over $(\mathbb{B},K)$:

$$^*S \equiv \mathrm{Hom}_{\mathbb{B}}(-,\hat{S}).$$

Thus, for instance, if $\mathscr{F}$ is the basis of the Fréchet filter, $^*S(\mathscr{F})$ will consist of infinite sequences of S-objects, where two such are considered equal if they as sequences agree eventually. A function $f : (S_1,=_1) \times \cdots \times (S_k,=_k) \to (S,=)$ yields a natural transformation $^*f : {}^*S_1 \times \cdots \times {}^*S_k \mathrel{\dot\to} {}^*S$ by defining

$$(^*f)_{\mathscr{F}}(\alpha_1,\ldots,\alpha_k) \equiv \lambda xv.(\exists s_1,\ldots,s_k) \bigwedge_{j=1}^{k} \alpha_j(x,s_j) \wedge f(s_1,\ldots,s_k) = v,$$

where $\mathscr{F} \equiv (I,J,=_J,F)$ is any filter base in $\mathbb{B}$. Composition of functions becomes composition of natural transformations. For any $c \in S$ there is a natural transformation ${}^*c : 1 \dot{\rightarrow} {}^*S$ given by

$$({}^*c)_{\mathscr{F}} \equiv \lambda z.(\lambda x \in J)(\lambda y \in S)\, y =_S c.$$

Let $R(s_1,\ldots,s_k)$ $(s_1 \in S_1,\ldots,s_k \in S_k)$ be any small relation which respects each of the equivalences $=_1,\ldots,=_k$. Define a $(\mathbb{B},K)$-relation on the sheaf ${}^*S_1 \times \cdots \times {}^*S_k$ as follows

$$(\alpha_1,\ldots,\alpha_k)\,\dot{\in}\,{}^*R(\mathscr{F}) \iff (\exists i)(\forall x \,\dot{\in}\, F_i)\,\exists(y_1,\ldots,y_k)\,\dot{\in}\, R \bigwedge_{j=1}^{k} \alpha_j(x,y_j), \tag{2.1}$$

for $\alpha_1 \in {}^*S_1(\mathscr{F}),\ldots,\alpha_k \in {}^*S_k(\mathscr{F})$. Recall that being a $(\mathbb{B},K)$-relation means that apart from preserving truth under restriction (monotonicity) also satisfying the cover property for K (local character).

3. The nonstandard model

In this section we build the nonstandard model of the (small) mathematical objects of type theory. Unlike the classical nonstandard models, e.g. [11], we do not understand semantics in the Tarskian sense. Instead we use sheaf semantics [3].

Define the first order language L of small sets with equality. This language will name all standard mathematical objects which belong to the universe U. The sorts are pairs $(S,=_S)$ consisting of a small set S and a small equivalence relation $=_S$. To this sort we associate the relation symbol $=_S$. A function symbol f of sort $(S_1,=_1)\times\cdots\times(S_k,=_k) \to (S,=)$ is a function $f : S_1 \times \cdots \times S_k \to S$ which respects the equalities. A relation symbol of sort $(S_1,=_1)\times\cdots\times(S_k,=_k)$ is a small relation R on $S_1 \times \cdots \times S_k$ which respects the equalities. A first order formula in this language has a canonical interpretation as a type-theoretic proposition. The nonstandard interpretation of L in $\mathrm{Sh}(\mathbb{B})$ is obtained by adding the $*$ to the left (cf. Section 2.1). Let $\Vdash$ be denote the forcing relation associated with this interpretation. To each relation R in L there is a $(\mathbb{B},K)$-relation ${}^*R$, as in (2.1) above. The forcing condition for this relation is then

$$\mathscr{F} \Vdash R(\alpha_1,\ldots,\alpha_n) \quad \Longleftrightarrow_{\mathrm{def}} \quad (\alpha_1,\ldots,\alpha_n)\,\dot{\in}\,{}^*R(\mathscr{F}).$$

We refer to a standard text (e.g. [3]) for the clauses of the logical constants, and the treatment of terms.

Note that in this interpretation not every sheaf is inhabited. To obtain a sound interpretation of intuitionistic logic we must take care to only use inhabited sorts, for the introduced variable in the introduction rule for $\exists$. This applies to the soundness theorem. Below we explicitly state when this precaution is necessary. The proof of the following fundamental theorem is analogous to that of Lemma 2.1 in [6], see also [7].

Theorem 3.1 (Moerdijk [6]). *Let $A(\bar{x})$ be an L-formula where $\bar{x} = x_1^{S_1}, \ldots, x_n^{S_n}$. Then at any filter base $\mathscr{F} = (I, =_I,\ J, =_J, F)$ and for any $\alpha_1 \in {}^*S_1(\mathscr{F}), \ldots, \alpha_n \in {}^*S_n(\mathscr{F})$ it holds that*

$$\mathscr{F} \Vdash A(\alpha_1, \ldots, \alpha_n) \quad \textit{iff} \quad (\exists i \in I)(\forall x \mathrel{\dot\in} F_i)(\exists y_1 \cdots y_n) \bigwedge_{i=1}^{n} \alpha_i(x, y_i) \wedge A(y_1, \ldots, y_n).$$

Corollary 3.2 (Transfer principle). *An L-sentence A is true if and only if $\Vdash A$.*

The sentence may contain arbitrary mathematical objects of the universe U as parameters, so this is indeed a useful transfer principle. Sometimes we use Robinson's $*$-notation to emphasise (some of) these parameters. The logic of the nonstandard model is weaker than the metatheory. In particular we cannot use the full axiom of choice except when it comes for free via the transfer principle.

3.1. Standard objects according to the model

For any set with equality $(S, =_S)$ we define a *standard predicate St^S* on the simple sheaf ${}^*S$ as follows

$$\beta \mathrel{\dot\in} \mathsf{St}^S(\mathscr{F}) \iff (\exists m)(\exists r_1, \ldots, r_m \in S)(\exists i)(\forall x \mathrel{\dot\in} F_i)[\beta(x, r_1) \vee \cdots \vee \beta(x, r_m)].$$

The right-hand side states that β takes at most finitely many values on some base set of the filter $\mathscr{F}$. In view of the definition of a cover, it means that β is "locally constant". It is straightforward to verify that St^S a $(\mathbb{B}, K)$-relation. The language L expanded with a predicate symbol St^S for every sort S is denoted by L', and the interpretation of St^S is naturally St^S. We use $\forall^{st} y^S \cdots$ as an abbreviation of $\forall y^S[St^S(y) \to \cdots]$, and $\exists^{st} y^S \cdots$ as a short form of $\exists y^S[St^S(y) \wedge \cdots]$; we often drop unnecessary sort information when it can be inferred from the context. If A is an L-formula, A^{st} denotes the formula obtained by restricting all quantifiers of A to standard objects. We have the following important result about standard quantifiers.

Lemma 3.3. *Let $A(\bar{x}, y)$ be any L'-formula. Then*:

(a) $\mathscr{F} \Vdash (\forall^{st} y^S)A(\bar{\alpha}, y)$ *iff for all $t \in S$, $\mathscr{F} \Vdash A(\bar{\alpha}, t)$,*

(b) $\mathscr{F} \Vdash (\exists^{st} y^S)A(\bar{\alpha}, y)$ *iff there are $t_1, \ldots, t_n \in S$ and a cover $\langle \beta_i : \mathscr{F}_i \to \mathscr{F} \rangle_{i=1}^n$ such that*

$$\mathscr{F}_i \Vdash A(\bar{\alpha} \circ \beta_i, t_i) \qquad (i = 1, \ldots, n).$$

Proof. Part (a) is analogous to Lemma 4.7 of [7]. The $(\Leftarrow)$ direction of part (b) is straightforward. As for the reverse direction, suppose

$$\mathscr{F} \Vdash (\exists^{st} y^S)A(\bar{\alpha}, y).$$

By the forcing semantics there is a cover $\langle \beta_i : \mathscr{F}_i \to \mathscr{F} \rangle_{i=1}^n$, and some maps $\gamma_i \in {}^*S(\mathscr{F}_i)$, $i = 1, \ldots, n$, such that

$$\gamma_i \mathrel{\dot\in} St^S(\mathscr{F}_i), \qquad \mathscr{F}_i \Vdash A(\bar{\alpha} \circ \beta_i, \gamma_i).$$

Thus there are constants $r^i_j \in S$ $(i = 1,\ldots,n,\ j = 1,\ldots,r^i_{m_i})$ with the property that for each i,

$$\exists p(\forall x \dot{\in} F_{i,p})[\gamma_i(x, r^i_1) \vee \cdots \vee \gamma_i(x, r^i_{m_i})].$$

Form the filter bases $\mathscr{G}_{i,j}$ by letting $x \dot{\in} G_{i,j,p}$ iff $x \dot{\in} F_{i,p}$ and $\gamma_i(x, r^i_j)$. Then each $\langle \mathscr{G}_{i,j} \hookrightarrow \mathscr{F}_i \rangle_{j=1}^{m_i}$ is a cover, so that the composition $\langle \beta_i : \mathscr{G}_{i,j} \to \mathscr{F} \rangle_{i,j}$ is a cover with the desired property. □

Note that, classically, if the terminal object $\mathscr{F} \equiv 1$ is covered by $\{\mathscr{F}^{(i)}\}$, then it is already covered by some $\mathscr{F}^{(i_0)}$. This is however not constructively true in general. Hence we cannot always conclude from $\Vdash \exists^{st} x\, A(x)$, that $\Vdash A(t_{i_0})$ for some t_{i_0}. But for certain classes of formulas it is indeed possible, see Theorem 3.6 below.

The standard predicate respects equality and is stable in the sense of intuitionistic logic.

Proposition 3.4. *Let S and T be sorts, and suppose that T has a decidable equality. Then the following statements are true in the model:*

(a) $\forall x^S\, y^S\, [St(x) \wedge x =_S y \to St(y)]$,

(b) $\forall x^T\, [\neg\neg St(x) \to St(x)]$.

Proof. Analogous to [6]. □

The application of functions can be expressed in two ways in the language L. For $f \in A \to B$ and $a \in A$ we have the interpretations ${}^*(Ap(f,a))$ and ${}^*Ap({}^*f, {}^*a)$. However, it is easy to see that $\Vdash {}^*(Ap(f,a)) = {}^*Ap({}^*f, {}^*a)$. An immediate consequence of Lemma 3.3 is

Corollary 3.5. $\Vdash (\forall^{st} f^{A \to B})(\forall^{st} a^A)\, St^B(Ap(f,a))$. □

In a classical metatheory it is straightforward to show that $\Vdash \forall^{st}\bar{x}[A^{st}(\bar{x}) \leftrightarrow A(\bar{x})]$, for any L-formula $A(\bar{x})$, so that by the transfer principle, the standard world of small objects is the same as the world defined by the St-predicate within the model (see [6]). In our constructive metatheory this is far from true, see e.g. (3.3). The following theorem yields some delineation of its truth. Let $\mathscr{H}$ be a class of formulas. A *positive combination* from this class is a formula constructed from $\mathscr{H}$-formulas using only the logical constants $\wedge$, $\vee$, $\forall$ and $\exists$. For example, a prenex formula is a positive combination of quantifier free formulas.

Theorem 3.6. *Let A and B be L-formulas involving only inhabited sorts.*

(a) *If A is a closed, positive combination of quantifier free formulas, and A is true, then $\Vdash A^{st}$.*

(b) *Suppose that $B(\bar{x}, \bar{y})$ has the property that for all $\bar{y}_1, \ldots, \bar{y}_n$, there exists $\bar{z}$ such that for all $\bar{x}$*

$$B(\bar{x}, \bar{y}_1) \vee \cdots \vee B(\bar{x}, \bar{y}_n) \Longrightarrow B(\bar{x}, \bar{z}).$$

Then: $\exists \bar{y} \forall \bar{x}\, B(\bar{x}, \bar{y})$ holds iff $\Vdash \exists^{st} \bar{y} \forall^{st} \bar{x}\, B(\bar{x}, \bar{y})$.

Proof. (a) By induction on the construction of A. The case where A is quantifier free follows immediately from the transfer principle. The cases for $\wedge$ and $\vee$ follows from the induction hypothesis. Using Lemma 3.3 and the inductive hypothesis, the cases for the quantifiers are direct.

(b, $\Rightarrow$) is immediate. As for (b, $\Leftarrow$) use Lemma 3.3, transfer and the property of B. □

3.2. The natural numbers of the model

The natural numbers of the model are very similar to those of a classical nonstandard model.

Proposition 3.7. *The following statements hold in the model:*
(a) $\exists x^{\mathbb{N}} \neg St^{\mathbb{N}}(x)$,
(b) $\forall \bar{x}\,(A(\bar{x},0) \wedge \forall^{st} y[A(\bar{x},y) \to A(\bar{x},S(y))] \longrightarrow \forall^{st} y\, A(\bar{x},y))$,
(c) $\forall x^{\mathbb{N}}\, y^{\mathbb{N}}[x < y \wedge St(y) \to St(x)]$,
(d) $\forall x^{\mathbb{N}}\,[\neg St(x) \leftrightarrow \forall^{st} y(y < x)]$.

Proof. (a) It can be easily be shown that for any filter base $\mathscr{F}$

$$\mathscr{F} \Vdash \neg St(\alpha) \quad \text{iff} \quad \forall m\, \exists i\, (\forall u \,\dot{\in}\, F_i)\, \alpha(u) \geqslant m \tag{3.1}$$

(see [6] or [7]). Let $\mathscr{F}$ be the base of the Fréchet filter on $\mathbb{N}$ given by $n \,\dot{\in}\, F_m \Leftrightarrow n \geqslant m$. Let α be the identity. Then clearly $\alpha : \mathscr{F} \to \hat{\mathbb{N}}$ is a morphism and $\mathscr{F} \Vdash \neg St(\alpha)$ holds by (3.1). The constant map $\beta : \mathscr{F} \to \mathscr{I}$ to the terminal object is a cover. Hence $\Vdash \exists x^N \neg St^N(x)$.

The *external induction schema* (b) follows readily from Lemma 3.3. The statements (c) and (d) follows from (b) and Theorem 3.1. □

This proposition shows first of all that there exists a nonstandard number and that, as in classical models, the standard numbers form an initial segment. By (d), a natural number is infinite precisely when it is nonstandard. The following result states that the nonstandard numbers have the order type of an unbounded dense set of copies of the integers. Define two relations $m \sim_s n$ iff $\exists^{st} k\,[m + k = n \vee n + k = m]$, and $m <_s n$ iff $\forall^{st} k\,(m + k < n)$.

Theorem 3.8. *The following hold in the model:*
(a) *$<_s$ is a transitive and dense ordering,*
(b) *$<_s$ is unbounded, i.e. $(\forall m)(\exists n)\, m <_s n$,*
(c) *$n \sim_s 0$ if and only if n is standard,*
(d) *for all m and n with $\neg m \sim_s n$, either $m <_s n$ or $n <_s m$.*

Proof. We work entirely within the model, and by the transfer principle we may use all first-order properties of natural numbers. In particular, the relations $<$ and $=$ are

decidable. The order $<_s$ is clearly transitive. As for density, assume $m <_s n$. If $m+n$ is even, let $\ell = (m+n)/2$. Thus $m < \ell < n$. For any standard k, we see that both $\ell \leqslant m+k$ and $n \leqslant \ell + k$ contradicts $m <_s n$. Hence $m <_s \ell <_s n$. The case when $m+n$ is odd, is similar. In this case let $\ell = (m+n+1)/2$.

To prove (b), we need just to note that $x <_s x+n$ for any infinite n. The equivalence (c) is obvious.

Finally, to prove (d) assume $\neg x \sim_s y$. Thus for all standard k, $x+k \neq y$ and $y+k \neq x$. From $x < y$, follows then by external induction that, $\forall^{st} k\,(x+k < y)$, i.e. $x <_s y$. Similarly from $y < x$, follows $y <_s x$. The case $x = y$ is impossible. □

3.3. The real numbers of the model

According to the transfer principle, the nonstandard real numbers inherits the first order properties of the standard real numbers expressible in the language L. But the relation between standard and nonstandard reals is different from the classical situation, most notably there is no standard part map (cf. [9]). Recall that two real numbers a and b are *infinitely close*, $a \simeq b$, if $\forall^{st} n\, |a-b| < 2^{-n}$. We have for positive rationals $q > 0$,

$$\not\Vdash \forall x\, [|x| < q \longrightarrow (\exists^{st} y)\, y \simeq x] \tag{3.2}$$

that is a finite real number is not necessarily infinitely close to a standard real number. To see this, let $\mathscr{H}$ be the trivial filter on $\mathbb{N}$, and let $\alpha : \mathbb{N} \longrightarrow \mathbb{R}$ be an enumeration of the rational numbers in the interval $(-q, q)$. Thus $\mathscr{H} \Vdash |\alpha| < q$. Suppose that $\mathscr{H} \Vdash (\exists^{st} y)\, y \simeq \alpha$. Hence there is a cover $\langle \varphi_i : \mathscr{F}_i \longrightarrow \mathscr{H} \rangle_{i=1,\ldots,n}$ and real numbers $r_1, \ldots, r_n$ such that $\mathscr{F}_i \Vdash r_i \simeq \alpha \circ \varphi_i$ for $i = 1, \ldots, n$. Let $\varepsilon > 0$ be so small that $n\varepsilon < q/2$. Thus $\mathscr{F}_i \Vdash |r_i - \alpha \circ \varphi_i| < \varepsilon$, and by the cover property it follows that every rational in the interval is within distance ε of some r_i. But this contradicts the choice of ε. Thus $\mathscr{H} \not\Vdash (\exists^{st} y)\, y \simeq x$.

There is a *Brouwerian counterexample* to the classical fact that

$$\Vdash \forall^{st} x\, (x \simeq 0 \longrightarrow x = 0). \tag{3.3}$$

The example involves a sequence (x_n) of rational numbers determining a real number x and a decidable filterbase F_n. Let $x_0 = 1$ and $F_0 = \{0\}$. We construct the sequence and the filter base inductively: x_{n+1} is half of x_n if no counterexample to Goldbach's conjecture has been found in $\{0, \ldots, n\}$, and $F_{n+1} = F_n$, but if we find a counterexample, we let $x_n = x_{n+1} = x_{n+2} = \cdots$, and $F_{n+1} = F_{n+2} = \cdots = \emptyset$. Then $\mathscr{F} \Vdash (x_n) \simeq 0$. If we had a constructive proof of (3.3), it would mean that $\mathscr{F} \Vdash (x_n) = 0$, i.e. for some k, we have $(x_n) = 0$ whenever the set F_k is nonempty. Thus by checking whether this set is nonempty we can decide Goldbach's conjecture.

Naturally, we do have a constructive proof that $x = 0$, whenever $\Vdash {}^*x \simeq 0$.

3.4. Saturation properties

The model is strongly saturated in the sense of the theorem below. We call an L-formula $A(\overline{x},\overline{y},\overline{p})$ *filtering in* $\overline{p}$, if for all $\overline{p}$ and $\overline{q}$ there exist some $\overline{r}$ such that

$$\forall \overline{x}\overline{y}\,[A(\overline{x},\overline{y},\overline{r}) \Longrightarrow A(\overline{x},\overline{y},\overline{p}) \wedge A(\overline{x},\overline{y},\overline{q})].$$

Theorem 3.9 (Saturation principle). *Let $A(\overline{x},\overline{y},\overline{p})$ be an L-formula involving only inhabited sorts. If this formula is filtering in $\overline{p}$, the following holds in the model:*

$$\forall \overline{x}\,[\forall^{st}\overline{p}\,\exists\,\overline{y}\,A(\overline{x},\overline{y},\overline{p}) \rightarrow \exists\,\overline{y}\,\forall^{st}\overline{p}\,A(\overline{x},\overline{y},\overline{p})].$$

Proof. Let $\overline{x} = x_1^{X_1},\ldots,x_m^{X_m}$, and let $\alpha_i \in {}^*X_i(\mathscr{F})$, $i=1,\ldots,m$, be such that

$$\forall \overline{p} \quad \mathscr{F} \Vdash \exists \overline{y} A(\alpha_1,\ldots,\alpha_m,\overline{y},\overline{p}).$$

For any $\overline{p}$ there is, by Theorem 3.1, some index w such that

$$\forall u \,\dot{\in}\, F_w\, \exists \overline{y}\, A(\alpha_1(u),\ldots,\alpha_m(u),\overline{y},\overline{p}). \tag{3.4}$$

Since A is a formula filtering in $\overline{p}$, we may define a filter $\mathscr{G}$ by

$$G_{w,\overline{p}} = \{(u,\overline{y}) : u \,\dot{\in}\, F_w \wedge A(\alpha_1(u),\ldots,\alpha_m(u),\overline{y},\overline{p})\}.$$

The first projection $\pi_1 : \mathscr{G} \rightarrow \mathscr{F}$ is a cover due to (3.4) and the fact that $\mathscr{F}$ is a filter. The other projections $\gamma_1,\ldots,\gamma_n$ into the respective sorts of $\overline{y}$ are trivially continuous. It is immediate, using Theorem 3.1, that

$$\forall \overline{p} \quad \mathscr{G} \Vdash A(\alpha_1 \circ \pi_1,\ldots,\alpha_m \circ \pi_1,\gamma_1,\ldots,\gamma_n,\overline{p}).$$

So by Lemma 3.3(a) and the fact that π_1 is a cover we get

$$\mathscr{F} \Vdash \exists \overline{y}\, \forall^{st}\overline{p}\, A(\alpha_1,\ldots,\alpha_m,\overline{y},\overline{p}). \qquad \square$$

As a remarkable consequence we note that the standard objects of any inhabited, standard set have a hyperfinite enumeration.

Corollary 3.10. *For any inhabited standard set $(S,=)$ (i.e. a sort in L) the following holds in the model*

$$\exists n^{\mathbb{N}}\, \exists g^{\mathbb{N}\rightarrow S}\, \forall^{st} a^S\, (\exists k < n)\, g(k) = a.$$

Proof. The formula

$$A(n,g,m,f) \equiv \exists k^{\mathbb{N}}\, [k+m<n \wedge (\forall p<m)\, g(k+p) = f(p)]$$

states that $f(0),\ldots,f(m-1)$ is a "substring" of $g(0),\ldots,g(n-1)$. It is easily seen to be filtering in m and f. Clearly, we have $\forall m\, \forall f\, \exists n\, \exists g\, A(n,g,m,f)$, so by Lemma 3.3 and Theorem 3.1

$$\Vdash \forall^{st} m\, \forall^{st} f\, \exists n\, \exists g\, A(n,g,m,f).$$

The saturation principle then yields

$$\Vdash \exists n\, \exists g\, \forall^{st} m\, \forall^{st} f\, A(n,g,m,f).$$

From this the result follows. □

In the model there is a dense scale of infinities (cf. the theories of J. Mycielski referred to in [8]). To see this we introduce an order relation on natural numbers: n is *inaccessible from* m, in symbols $m \ll n$, if $(\forall^{st} f)\,[f(m) < n]$. Note that in the model $0 \ll n$ is the same as $\neg St(n)$. An analogue of Theorem 3.8 holds for the inaccessibility relation.

Corollary 3.11. *In the model the relation* $\ll$ *is transitive, dense and unbounded.*

Proof. We work within the model. Transitivity is obvious. To prove density, first observe that $m \ll n$ is the same as $(\forall^{st} f)\, \sum_{k=0}^{m} f(k) < n$. It is easy to prove that the formula

$$A(f,m,\ell,n) \equiv \sum_{k=0}^{m} f(k) < \ell \wedge \sum_{k=0}^{\ell} f(k) < n,$$

is filtering in f by considering pointwise addition of functions. Suppose that $m \ll n$. Let f be any standard function. Let $g(x) = \sum_{k=0}^{x} f(k)$. Then $g(m) < g(m) + 1$ and $g(g(m) + 1) < n$. Hence for every standard f, the number $\ell = g(m) + 1$ satisfies $A(f,m,\ell,n)$. Thus by the saturation principle, there is some ℓ such that for all standard f, $A(f,m,\ell,n)$, i.e. $m \ll \ell$ and $\ell \ll n$.

To prove unboundedness consider the filtering formula $B(f,m,n) \equiv \sum_{k=0}^{m} f(k) < n$. We leave the details to the reader. □

Introduce the abbreviations $\forall^{inf} x^{\mathbb{N}} \cdots$ for $\forall x^{\mathbb{N}}[\neg St(x) \rightarrow \cdots]$, and $\exists^{inf} x^{\mathbb{N}} \cdots$ for $\exists x^{\mathbb{N}}[\neg St(x) \wedge \cdots]$.

Example 3.12. Using the saturation principle, one can show *Robinson's sequential lemma*:

$$\Vdash \forall a^{\mathbb{N}\to\mathbb{R}}\, [\forall^{st} n\, a_n \simeq 0 \rightarrow \exists^{inf} m\, (\forall n \leqslant m)\, a_n \simeq 0].$$

The axiom of choice seems to hold only to a very limited extent. The following is also a corollary to the saturation principle.

Corollary 3.13. *Let* $A(\bar{u}, n, y^S)$ *be an L-formula involving only inhabited sorts. Then*

$$\Vdash \forall \bar{u}\, [\forall^{st} n^{\mathbb{N}}\, \exists y^S\, A(\bar{u}, n, y) \longrightarrow \exists f^{\mathbb{N}\to S}\, \forall^{st} n^{\mathbb{N}}\, A(\bar{u}, n, f(n))].$$

Proof. In the model first prove that $(\forall^{st} n)(\exists f)(\forall i < n) A(\bar{u}, i, f(i))$ by using the external induction schema (Proposition 3.7(b)). Then note that the formula $(\forall i < n)$

$A(\bar{u}, i, f(i))$ is filtering in n. Hence by the saturation principle

$$(\exists f)(\forall^{st} n)(\forall i < n) A(\bar{u}, i, f(i)),$$

which yields the conclusion. $\square$

Remark 3.14. Shortly after [6] was written, Moerdijk observed that his model construction and the Łos-type Theorem 3.1 goes indeed through for any first order structure. The main difference between his construction and ours is that we use filter bases instead of filters, and that we explicitly work within a constructive and *predicative* framework. In most cases the proofs translate easily from filters to filter bases. Some further examples of this translation can be found in [7], where a syntactic version of filter bases is used.

Theorems 3.6 and 3.8 and Theorem 3.9 with its corollaries are new results about the model. The examples of Section 3.3 are also new.

Even though the metatheory of this model can be taken to be classical set theory, the logic of the model remains nonclassical. We cite [6]:

Theorem 3.15. $\Vdash \neg\forall x^{\mathbb{N}}(St(x) \vee \neg St(x))$. $\square$

4. Nonstandard characterisations of notions in analysis

In our earlier approach to constructive nonstandard analysis [8, 9] we used a reduced power model, where the reduction was taken by the Fréchet filter. This approach was strongly influenced by works of D. Laugwitz, C. Schmieden, and P. Martin-Löf. We could successfully give nonstandard characterisations of notions involving simple sequential limits. (The first characterisation of a simple limit in the sheaf model was given by Moerdijk.) However, one feature of constructive analysis is that many notions cannot, as in classical analysis, be reduced to such limits, e.g. uniform continuity on a compact interval, total boundedness. In Sections 4.1, 4.2 and 4.3 below, it will be seen that by employing the rich filter structure of the sheaf model we can constructively regain some of the classical characterisations, known from Robinson [11]. We also obtain a characterisation of iterated limits (Theorem 4.4) which we have not seen in the classical setting.

4.1. Modes of convergence

First some results about convergence of double sequences in a fixed metric space (X, d). On this space the relation $a \simeq b$ means that $(\forall^{st} n)\, d(a, b) < 2^{-n}$.

Theorem 4.1. *Let $a : \mathbb{N} \times \mathbb{N} \to X$ be a double sequence, and let $b : \mathbb{N} \to X$ be a sequence.*

(a) *The double sequence $(a_{m,n})$ converges to $c \in X$ if, and only if,*

$$\Vdash (\forall^{inf} m, n)\, [{}^* a_{m,n} \simeq {}^* c].$$

(b) *The double sequence* $(a_{m,n})$ *converges to the sequence* (b_m) *uniformly in* m *if, and only if,*

$$\Vdash (\forall^{inf} n)(\forall m)\,[{}^*a_{m,n} \simeq {}^*b_m].$$

Proof. (a, $\Rightarrow$) Let $\mu, \nu \in {}^*\mathbb{N}(\mathscr{F})$ be such that $\mathscr{F} \Vdash \neg St(\mu)$ and $\mathscr{F} \Vdash \neg St(\nu)$. Take an arbitrary $k \in \mathbb{N}$. By the assumption there is a p such that $d(a_{m,n}, c) < 2^{-k}$ for all $m, n \geqslant p$. By Proposition 3.7(d) and the fact that $\mathscr{F}$ is a filter base, there exists q such that $\mu(u), \nu(u) \geqslant p$ for all $u \in F_q$. Thus $\mathscr{F} \Vdash d(a_{\mu,\nu}, c) < 2^{-k}$.

(a, $\Leftarrow$) For any $k \in \mathbb{N}$ we have $\Vdash (\forall^{inf} m, n)\,[d(a_{m,n}, c) < 2^{-k}]$. Consider the "square" $\mathscr{C}$ of the Fréchet filter base:

$$C_p \equiv \{(m,n) \in \mathbb{N}^2 : m, n \geqslant p\}.$$

It is straightforward to see that the projections $\pi_i : \mathscr{C} \to \mathbb{N}$ $(i=1,2)$ satisfy $\mathscr{C} \Vdash \neg St(\pi_i)$, so $\mathscr{C} \Vdash d(a_{\pi_1,\pi_2}, c) < 2^{-k}$. Hence for some $p \in \mathbb{N}$

$$(\forall m, n \geqslant p)\,[d(a_{m,n}, c) < 2^{-k}].$$

(b, $\Rightarrow$) Left to the reader.

(b, $\Leftarrow$) Let k be such that $\Vdash (\forall^{inf} n)(\forall m)\,[d(a_{m,n}, b_m) < 2^{-k}]$. Now we evaluate this statement at the trivial filter times the Fréchet filter, or more precisely at

$$G_p \equiv \{(m,n) \in \mathbb{N}^2 : n \geqslant p\}.$$

For the projections $\pi_1, \pi_2 : \mathscr{G} \to \mathbb{N}$ we thus have

$$\mathscr{G} \Vdash d(a_{\pi_1,\pi_2}, b_{\pi_1}) < 2^{-k}.$$

Hence there is some p such that $d(a_{m,n}, b_m) < 2^{-k}$ for $n \geqslant p$ and any m. $\square$

Example 4.2. A sequence $a : \mathbb{N} \to X$ is a Cauchy sequence if, and only if,

$$\Vdash (\forall^{inf} \mu, \nu)\, {}^*a_\mu \simeq {}^*a_\nu.$$

The following result is a straightforward generalisation of Theorem 4.1(b).

Theorem 4.3. *Let* (f_n) *be a sequence of functions from a set* U *into* X. *Then* (f_n) *converges uniformly to* $f : U \to X$ *if, and only if,*

$$\Vdash (\forall^{inf} n)(\forall u)[{}^*f_n(u) \simeq {}^*f(u)].$$

To treat iterated limits we use the inaccessibility relation $\ll$ defined in Section 3.

Theorem 4.4. *Let* $a : \mathbb{N} \times \mathbb{N} \to X$ *be a double sequence, and suppose that* $\lim_n a_{m,n}$ *exists for each* m. *Then* $\lim_m \lim_n a_{m,n} = c$ *iff*

$$\Vdash \forall mn[0 \ll m \ll n \to {}^*a_{m,n} \simeq {}^*c].$$

Proof. Observe that $\lim_m \lim_n a_{m,n} = c$ is equivalent to

$$\forall k\, \exists M\, \exists g\, (\forall m \geqslant M)(\forall n \geqslant g(m))\,[d(a_{m,n}, c) < 2^{-k}]. \tag{4.1}$$

The direction ($\Rightarrow$) is proved as follows. By the transfer principle, we have for each k some M and g such that

$$\Vdash (\forall m \geqslant {}^*M)(\forall n \geqslant {}^*g(m))\,[d(a_{m,n}, c) < 2^{-k}].$$

From $\mathscr{F} \Vdash 0 \ll \mu \ll \nu$ follows then immediately that $\mathscr{F} \Vdash d(a_{\mu,\nu}, c) < 2^{-k}$. Since k was arbitrary, in fact also $\mathscr{F} \Vdash a_{\mu,\nu} \simeq c$.

As for the direction ($\Leftarrow$) consider a filter base indexed by natural numbers M and functions $g \in \mathbb{N} \to \mathbb{N}$:

$$H_{M,g} \equiv \{(m,n) \in \mathbb{N}^2 \colon m \geqslant M, n \geqslant g(0) + \cdots + g(m)\}.$$

Let π_1 and π_2 be the first and second projections, respectively. It is easy to check that $\mathscr{H} \Vdash 0 \ll \pi_1 \ll \pi_2$. Hence we have that for any k

$$\mathscr{H} \Vdash d(a_{\pi_1,\pi_2}, c) < 2^{-k}.$$

From this (4.1) follows easily. □

To verify the characterisation of limit points is left to the reader.

Proposition 4.5. *Let $a \colon \mathbb{N} \to X$ be a sequence whose terms are all apart from L. Then L is a limit point of the sequence if and only if $\Vdash (\exists^{inf} \nu)\, {}^*a_\nu \simeq {}^*L$.*

4.2. Uniform continuity and differentiability

We recall that the basic notion of continuity in constructive analysis is uniform continuity [1]. Ordinary continuity is then defined as local uniform continuity. As in classical nonstandard analysis we have the characterisation:

Theorem 4.6. *Let (X,d) and (X',d') be metric spaces. A function $f \colon X \to X'$ is uniformly continuous if, and only if,*

$$\Vdash \forall u^X v^X\, [u \simeq v \to {}^*f(u) \simeq {}^*f(v)].$$

Proof. ($\Rightarrow$) Let $\alpha, \beta \in {}^*X(\mathscr{F})$ be such that $\mathscr{F} \Vdash \alpha \simeq \beta$. Take an arbitrary $n \in \mathbb{N}$. By the assumption there is some k with $(\forall u, v \in X)[d(u,v) < 2^{-k} \Rightarrow d'(f(u), f(v)) < 2^{-n}]$. Hence there is some p, such that $d(\alpha(w), \beta(w)) < 2^{-k}$ for all $w \mathrel{\dot\in} F_p$, and consequently

$$d'(f(\alpha(w)), f(\beta(w))) < 2^{-n} \quad (w \mathrel{\dot\in} F_p).$$

This shows that $\mathscr{F} \Vdash f(\alpha) \simeq f(\beta)$, since n was arbitrary.

($\Leftarrow$) Consider the filter base $\mathscr{U}$ given by

$$U_p \equiv \{(u,v) \in X^2 : d(u,v) < 2^{-p}\}.$$

It is easily seen that the projections $\pi_1, \pi_2 : \mathscr{U} \to X$ satisfy $\mathscr{U} \Vdash \pi_1 \simeq \pi_2$. Therefore $\mathscr{U} \Vdash f(\pi_1) \simeq f(\pi_2)$, which means that f is uniformly continuous. $\square$

Example 4.7. Using the above results, there is an easy proof that if (f_n) is a sequence of uniformly continuous functions $X \to X'$, converging uniformly to $f : X \to X'$, then f is uniformly continuous. The proof goes by reasoning as follows in the model: Let $x \simeq y$. Then for all standard n, ${}^*f_n(x) \simeq {}^*f_n(y)$. By Robinson's sequential lemma (Example 3.12), there exists an infinite v such that second member of

$${}^*f(x) \simeq {}^*f_v(x) \simeq {}^*f_v(y) \simeq {}^*f(y)$$

holds. The first and last member follows by uniform convergence.

To form subspaces the following constructions are useful. Let $(X, =_X)$ be a set with a (construed) equality. Any predicate P on X which respects $=_X$ defines a new set $\tilde{P} \equiv (\Sigma x \in X)P(x)$. The projection $i : \tilde{P} \to X$ defines a natural equality on $\tilde{P}$:

$$u =_{\tilde{P}} v \iff i(u) =_X i(v).$$

Other relations on X are inherited in a similar way by $\tilde{P}$. We have for

$$P(x) \iff (\exists v \in \tilde{P})x =_X i(v).$$

Let $I \dot{\subseteq} \mathbb{R}$ be a compact interval. Let $f, g \in \tilde{I} \to \mathbb{R}$ be uniformly continuous. Then g is the *derivative* of f if for every k there is an n such that

$$\forall x, y \in \tilde{I}\,[|x-y| \leqslant 2^{-n} \Longrightarrow |f(x) - f(y) - g(x)(x-y)| \leqslant 2^{-k}|x-y|].$$

Since the conclusion of this implication consists of continuous functions, it is enough to check the condition for $x \neq y$. We may thus rewrite the implication as

$$\forall x, y \in \tilde{I}\, \forall p \in \mathbb{N}\,[2^{-p} < |x-y| \leqslant 2^{-n} \Longrightarrow \left|\frac{f(x)-f(y)}{x-y} - g(x)\right| \leqslant 2^{-k}]. \quad (4.2)$$

Note that division is really a function of a p too, witnessing that the denominator is nonzero, though this is supressed in the notation. We obtain the characterisation of the derivative familiar from classical nonstandard analysis.

Theorem 4.8. *Let $\tilde{I}$ be a compact interval, and let $f, g : \tilde{I} \to \mathbb{R}$ be uniformly continuous. Then g is the derivative of f if, and only if,*

$$\Vdash \forall x^{\tilde{I}}, y^{\tilde{I}}, p^{\mathbb{N}}\,[2^{-p} < |x-y| \simeq 0 \to \frac{{}^*f(x) - {}^*f(y)}{x-y} \simeq {}^*g(x)].$$

Proof. This is analogous to the case of uniform continuity, but for the direction ($\Leftarrow$) we use instead the filter base $V_n \equiv \{(u,v,p) \in \tilde{I} \times \tilde{I} \times \mathbb{N} : 2^{-p} < |u-v| < 2^{-n}\}$. $\square$

Example 4.9 (*The product rule for differentiation*). Suppose that f and g have derivatives f' and g', respectively. Let $h(x) = f(x)g(x)$. Then we prove that $h' = f'g + fg'$ by reasoning in the model: for $x^{\tilde{I}}$, $y^{\tilde{I}}$ and $|x - y| > 2^{-p}$,

$$\frac{h(x)-h(y)}{x-y} = \frac{f(x)g(x)-f(y)g(y)}{x-y} = f(x)\frac{g(x)-g(y)}{x-y} + \frac{f(x)-f(y)}{x-y}g(y).$$

Using Theorem 4.8 we obtain for $x \simeq y$,

$$\frac{h(x)-h(y)}{x-y} \simeq f(x)g'(x) + f'(x)g(y) \simeq f(x)g'(x) + f'(x)g(x).$$

The last member follows by the uniform continuity of g. □

4.3. Open, closed and totally bounded sets

The characterisation of open sets is also familiar from the classical setting.

Theorem 4.10. *Let (X, d) be a metric space. Let $A \dot{\subseteq} X$, and let $i : \tilde{A} \to X$ be the canonical projection. Then A is an open set if, and only if,*

$$\Vdash (\forall x^{\tilde{A}})(\forall y^X)[i(x) \simeq y \to y \dot{\in} A].$$

Proof. ($\Rightarrow$) Let $a \in \tilde{A}$ and $\beta \in {}^*X(\mathscr{F})$ be such that $\mathscr{F} \Vdash i(a) \simeq \beta$. Since A is open there is an open ball $B_{2^{-k}}(i(a)) \dot{\subseteq} A$. But $\mathscr{F} \Vdash d(i(a), \beta) < 2^{-k}$, so there is some p such that for all $u \dot{\in} F_p$, it holds that $d(i(a), \beta(u)) < 2^{-k}$, and particular $\beta(u) \dot{\in} A$. Hence $\mathscr{F} \Vdash \beta \dot{\in} A$.

($\Leftarrow$) For each given $a \in \tilde{A}$ define a (neighbourhood) filter $\mathscr{F}^a$ by

$$F_p^a = \{v \in X : d(i(a), v) < 2^{-p}\}.$$

Then if β is the identity we have, trivially, $\mathscr{F}^a \Vdash i(a) \simeq \beta$. Hence $\mathscr{F}^a \Vdash \beta \dot{\in} A$. But this is equivalent to saying that $B_{2^{-p}}(i(a)) \dot{\subseteq} A$, for some p. □

However, the expected characterisation of a closed set A,

$$\Vdash \forall^{st} x^X \, \forall y^{\tilde{A}} \, [x \simeq i(y) \to x \dot{\in} A]$$

fails. It is nevertheless a sufficient condition. (In a classical metatheory it is also necessary condition.)

The characterisation of a totally bounded sets is that every element of the set is arbitrarily close to a standard element of the set. More precisely:

Theorem 4.11. *Let (X, d) be a metric space and let $A \dot{\subseteq} X$. Then A is totally bounded if, and only if,*

$$\Vdash \forall^{st} n \, \forall x^{\tilde{A}} \, \exists^{st} y^{\tilde{A}} \, [d(i(x), i(y)) < 2^{-n}].$$

Proof. ($\Rightarrow$) Let $\xi \in [\![\tilde{A}]\!](\mathscr{F})$ and n be arbitrary. We want to show that

$$\mathscr{F} \Vdash (\exists^{st} y^{\tilde{A}})\, d(i(\xi), i(y)) < 2^{-n}.$$

Let $b_1, \ldots, b_m \dot{\in} A$ be a 2^{-n}-net for $\tilde{A}$. Define filter bases

$$F_p^k \equiv \{u \in F_p : d(i(b_k), i(\xi(u))) < 2^{-n}\} \quad (k = 1, \ldots, m).$$

Thus $\langle \mathscr{F}^k \hookrightarrow \mathscr{F} \rangle_{k=1}^m$ is a cover, and clearly for each $k = 1, \ldots, m$,

$$\mathscr{F}^k \Vdash d(i(b_k), \xi) < 2^{-n}$$

The desired conclusion follows by Lemma 3.3.

($\Leftarrow$) Let n be arbitrary. Then $\tilde{A} \Vdash (\exists^{st} y^{\tilde{A}})[d(i(y), i(\eta)) < 2^{-n}]$, where $\tilde{A}$ is considered as a trivial filter base, and η is the identity. Hence there is a cover $\langle \beta_k : \mathscr{F}^k \to \tilde{A} \rangle_{k=1}^m$ and some constants $c_1, \ldots, c_m \in \tilde{A}$ such that for each k,

$$\mathscr{F}^k \Vdash d(i(c_k), i(\eta \circ \beta_k)) < 2^{-n}.$$

Thus there are indices $p_1, \ldots, p_m$ such that $d(i(c_k), i(\beta_n(u))) < 2^{-n}$ for $u \dot{\in} F_{p_k}^k$ and $k = 1, \ldots, m$. From the fact that $\langle \beta_k \rangle_{k=1}^m$ that is a cover it follows that for each $x \in \tilde{A}$ there is some k and some $u \dot{\in} F_{p_k}^k$ so that $\beta_k(u) =_{\tilde{A}} x$. This shows that $c_1, \ldots, c_m$ is a 2^{-n}-net for A. $\square$

Related work. Filters were introduced by H. Cartan to give a treatment of convergence, which was later expounded in detail by Bourbaki. Sheaf-theoretic models of synthetic differential geometry containing both invertible and nilpotent infinitesimals were studied by Moerdijk and Reyes [5]. Their models build on sites of smooth functions. The models were however not developed within a constructive framework. Fourman [2] constructs a site of all formal spaces. Sheaves over this site was considered for the purpose of modelling potentially infinite objects.

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References

[1] E. Bishop and D.S. Bridges, Constructive Analysis (Springer, Berlin, 1985).

[2] M.P. Fourman, Continuous truth I: nonconstructive objects, in: G. Lolli et al. eds., Logic Colloquium '82 (North-Holland, Amsterdam, 1984).

[3] S. Mac Lane and I. Moerdijk, Sheaves in Geometry and Logic (Springer, Berlin, 1992).

[4] P. Martin-Löf, Intuitionistic Type Theory (Bibliopolis, Naples, 1984).

[5] I. Moerdijk and G.E. Reyes, Models of Smooth Infinitesimal Analysis (Springer, Berlin, 1991).
[6] I. Moerdijk, A model for intuitionistic non-standard arithmetic, Ann. Pure Appl. Logic 73 (1995) 37–51.
[7] I. Moerdijk and E. Palmgren, Minimal models of Heyting arithmetic, J. Symbolic Logic, to appear.
[8] E. Palmgren, A constructive approach to nonstandard analysis, Ann. Pure Appl. Logic 73 (1995) 297–325.
[9] E. Palmgren, Constructive nonstandard analysis, in: A. Pétry ed., Méthodes et Analyse Non Standard, Cahiers du Centre de Logique, Vol. 9 (Academia, Louvain-la-Neuve, 1996).
[10] E. Palmgren, Constructive sheaf semantics, Math. Logic Quarterly, to appear.
[11] A. Robinson, Nonstandard Analysis (North-Holland, Amsterdam, 1966).