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The integral homology and cohomology rings of SO(n) and Spin(n)

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Abstract

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We give explicit generators and relations for the ring structure of the integral rings in the title. These results are new, at least for large n, and they complete the program of computing these rings for the classical groups.

Introduction

Let $G = \mathrm{SO}(n)$ or $\mathrm{Spin}(n)$; we will describe the integral rings $H^*(G : \mathbb{Z})$ and $H_*(G : \mathbb{Z})$ (under Pontryagin product) in terms of explicit generators. Despite the considerable literature on this subject, some of it dating back to the 1950's, the multiplicative structure was not previously known except for small values of n (see, for example, the introduction to the recent paper of Kač [4]). In fact, even $H^*(\mathrm{SO}(n)/T : \mathbb{Z})$, where T is a maximal torus in SO(n), was computed only in the 1970's [6, 10]. We shall give a simple computation of this last ring and then use the spectral sequence of $T \to G \to G/T$ to get at G; along the way we answer affirmatively a question of Kač [4] proving that the *integral* Serre spectral sequence of this bundle for $G = \mathrm{SO}(n)$ or Spin(n) collapses at E_3. This is the main reason for using the Serre spectral sequence in the paper. We are also well-nigh forced to give a new computation of $H^*(\mathrm{Spin}(n) : \mathbb{F}_2)$ as an algebra over the Steenrod algebra $\mathscr{A}(2)$; this was done by Borel [3] only on the transgressive part, and subsequently completed to the 'exceptional generator' in an announcement

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by Svarč [8]. However, our definition of this exceptional generator and computations with it are more global and completely free of cell-decompositions.

Of course, the structure of $H^*(G:\mathbb{Q})$ and $H_*(G:\mathbb{Q})$ is classical from Hopf's theorem. Miller [7] computed $H^*(\mathrm{SO}(n):\mathbb{F}_2)$ as a module over $\mathscr{A}(2)$ and proved that the torsion subgroup of $H^*(\mathrm{SO}(n):\mathbb{Z})$ has exponent 2, thus giving the *additive* structure of the integral cohomology; he also computed $H_*(\mathrm{SO}(n):\mathbb{F}_2)$. Borel [3] derived these results on $\mathrm{SO}(n)$ concisely from his celebrated Transgression Theorem, and extended some of them to $\mathrm{Spin}(n)$. In [3] he computed $H^*(\mathrm{Spin}(n):\mathbb{F}_p)$, $p\geq 2$ and enough of the $\mathscr{A}(2)$ module structure of $H^*(\mathrm{Spin}(n):\mathbb{F}_2)$ to conclude that the torsion subgroup of $H^*(\mathrm{Spin}(n):\mathbb{Z})$ has exponent 2; he also computed $H_*(\mathrm{Spin}(n):\mathbb{F}_2)$ for $n\leq 10$ and proved the noncommutativity of this ring for $n=10$. The description of the Pontryagin product in $H_*(\mathrm{Spin}(n):\mathbb{F}_2)$ was completed by Svarč [8]; and apparently this is also due to Kojima [5] (but I am relying on later papers of S. Araki and W. Browder for this attribution, not having access to Kojima's paper).

Nontopologists to the contrary, the cup-product in $H^*(G:\mathbb{F}_p)$ (for all $p\geq 2$) does *not* give the integral cup-product, even in the presence of the additive description of the integral cohomology; the point is that certain elements of infinite order have squares (and higher powers) of order 2, and this cannot be deduced simply from the modular rings. What one has to do is to create enough integral elements, in a sense a priori, to generate the integral ring; then one can compute the relations (as well as the Pontryagin product) using the results of Miller and Borel, by relating the mod 2 reductions of the integral generators to an explicit set of generators for $H^*(G:\mathbb{F}_2)$. It is thus to match our generators with known computations that we have to rederive known results in our own terms; this accounts largely for the length of the paper.

The results of this paper complete the description of the integral cohomology and Pontryagin rings for all classical simply-connected groups. The exceptional groups, $\mathrm{PSO}(2n)$ and the 'semi-Spin' groups $\mathrm{SSpin}(4n)$ will be taken up elsewhere.

It turns out that the calculations for $\mathrm{SO}(n)$ are parallel to those of $\mathrm{Spin}(n)$ and somewhat simpler; thus $\mathrm{SO}(n)$ is relegated to the last section of this paper where the parallels are briefly set out. Similarly, $\mathrm{Spin}(2n)$ shares many features in common with $\mathrm{Spin}(2n+1)$; thus the main focus of the paper is on $\mathrm{Spin}(2n+1)$, with occasional remarks on the case of type D_n. In fact, using certain general features of the computations of B_n and D_n—and without recourse to the nitty-gritty of the full description—we prove (in Section 2) the following theorem:

Theorem 1. *There is an integral class ε in $H^{2n+1}(\mathrm{Spin}(2n+2):\mathbb{Z})$ giving a ring isomorphism*

$$H^*(\mathrm{Spin}(2n+2):\mathbb{Z})\simeq H^*(\mathrm{Spin}(2n+1):\mathbb{Z})\otimes\Lambda^*(\varepsilon)\,.$$

(Of course, ε transgresses to the Euler class in the universal bundle.) The

corresponding result is true for SO(n) as well, and the theorem reduces our task to establishing only those features of type D_{n+1} which are needed for this proof. However, the analogous assertion is *false* for the Pontryagin product in the homology of Spin($2n+2$): see Theorem 3 in Section 6.

Here is a brief description of $H^*(\mathrm{Spin}(2n+1):\mathbb{Z})$. There are three sets of generators; $x_1,\dots,x_n$, u_{2j-1} $(2\le j\le[(n+1)/2])$ and v_I where I is a certain multi-index (see Section 5). The elements x_j have degree $4j-1$ and infinite order, and their square-free products form a $\mathbb{Z}$-basis for the free part of the cohomology ring. The other generators are of order 2. The u_i generate the so-called 'Chow ring' and the v_I come from the Bockstein of a distinguished set of elements in $H^*(\mathrm{Spin}(2n+1):\mathbb{F}_2)$. All squares land in the Chow ring, and in fact one can eliminate roughly half the u_k by using the squares of the x_j (see Proposition 5.5). The rather long list of relations among these generators can be found in Propositions 5.5–5.8.

The principal effort in this paper is concentrated on the following:

(a) The construction of explicit elements $x_1,\dots,x_n$ with the properties above; this is taken up in Sections 4 and 5 using the results of Section 3 on $H^*(\mathrm{SO}(n)/T:\mathbb{Z})$.

(b) The proof in Section 4 that the torsion ideal in $E_3(\mathbb{Z})$ for

$$T\to \mathrm{Spin}(n)\to \mathrm{Spin}(n)/T$$

is an $\mathbb{F}_2$-vector space; together with (a) this is the crucial step in proving $E_3(\mathbb{Z})=E_\infty(\mathbb{Z})$ because, unlike the modular case [4], as soon as $n\ge 5$ there are generators in $E_3(\mathbb{Z})$ of fibre degree ≥ 2.

(c) The isolation of the exceptional generator of $H^*(\mathrm{Spin}(n):\mathbb{F}_2)$ and the action of $\mathscr{A}(2)$ on it which also uses steps from (a). With the aid of (a) and (c) and an argument which we hope the reader will not find too devious, we can prove Theorem 1 (Section 2). In Section 6 we work out the integral Pontryagin product for Spin(n) and in Section 7 we record the parallel assertions (of Sections 4–6) for SO(n).

1. Notations and preliminaries

1.1. R will denote the coefficient ring for all homology and cohomology groups in this paper; it will always be a *principal ideal domain*, in practice being $\mathbb{Z}$, a localization of $\mathbb{Z}$ (including $\mathbb{Q}$) or one of the Galois fields $\mathbb{F}_p$.

1.2. Let A^* be a (graded) R-algebra. A set of (homogeneous) elements $a_1,\dots,a_n$ is said to be a *simple set of generators* for A^* iff the set

$$\{1, a_{i_1}\cdots a_{i_p} \mid 1\le i_1<\cdots<i_p\le n\}$$

is an R-basis for A^*. In this case we write

$$A^* = \Delta^*(a_1, \ldots, a_n) .$$

Note that in particular A^* is connected if $\deg a_i > 0$ for all i and a free R-module of rank 2^n; this terminology differs from that of Borel [3] when char $R \neq 2$.

1.3. If M is a closed oriented manifold, we will denote by $[M] \in H^{\text{top}}(M : R)$ the generator Poincaré-dual to the class of a point in $H_0(M : R)$ and call it the *fundamental class*.

We now list some trivial homological facts which will be repeatedly used in the following sections.

1.4. Suppose (C^*, d) is a free R-cochain complex of finite type; if $\mathrm{rk}_R C^* = \mathrm{rk}_R H^*$, then $d = 0$ and $C^* = H^*$. Thus the fact that H^* is a free R-module is part of the conclusion. This assertion will be used repeatedly to prove the degeneration of various spectral sequences.

1.5. If $\varphi : F \to E \to B$ is a fibre-bundle with connected fibres, the Serre spectral sequence of φ with coefficients in R is denoted by $E_*(\varphi : R)$. If $R \to S$ is a flat extension, then $E_r(\varphi : S) = E_r(\varphi : R) \otimes_R S$, $d_r^S = d_r^R \otimes_R S$.

1.6. There is a fairly simple formal 'Poincaré-duality' argument which is used several times in the paper, so we given an abstract formulation here. The proof can be found in 4.5.

Suppose A^* is a finitely-generated P-D algebra with fundamental class $[A]$ defined over the PID R. Let $a_1, \ldots, a_r \in A^*$ be elements such that

(i) $a_j^2 = 0$, $a_j a_k = -a_k a_j$,

(ii) $a_1 \cdots a_r = [A]$.

Then the subalgebra generated by $a_1, \ldots, a_r$ is an exterior algebra of rank 2^r and is an R-summand of A^*.

If, moreover, A^* is equipped with a derivation δ of square zero, $[A] \not\subseteq \mathrm{Im}\, \delta$ and $\delta(a_j) = 0$, then the exterior algebra $\Lambda^*(a_1, \ldots, a_r)$ intersects $\mathrm{Im}\, \delta$ in (0) and hence injects onto a summand of $H^*(A, \delta)$.

The end of a proof is indicated by a square '□'. A bar over an element of a ring denotes its reduction mod 2.

2. A 'reduction' to the odd case

As explained in the Introduction, we will prove Theorem 1 by culling certain general features of $H^*(\mathrm{Spin}(n) : \mathbb{Z})$ which are established in later sections.

Although this does not completely eliminate the consideration of Spin($2n$) from later sections, it does simplify our task somewhat. The analogous result for SO(n) is simpler and can be proved along the same lines. The reader might prefer to return to this section after going through Sections 3–5.

For a positive integer m define

$$\lambda(m) = \min\{d \mid m \le 2^d\}\,.$$

The following results are due to Borel [3, Section 12]:

2.1. (a) *If* 2 *is invertible in* R, $H^*(\mathrm{Spin}(m):R)$ = *exterior algebra on* $[m/2]$ *generators of odd degree.*

(b) $\dim_{\mathbb{F}_2}(H^*(\mathrm{Spin}(m):\mathbb{F}_2)) = 2^{m-\lambda(m)}$.

(c) *The* 2-*torsion subgroup of* $H^*(\mathrm{Spin}(m):\mathbb{Z})$ *has exponent* 2.

Now consider the fibre bundle

$$\omega_n\colon \mathrm{Spin}(2n+1)\to\mathrm{Spin}(2n+2)\xrightarrow{\pi} S^{2n+1}$$

and for the duration of this section put $G_n = \mathrm{Spin}(2n+1)$, $G_{n+1} = \mathrm{Spin}(2n+2)$.

2.2. Proposition. *For any coefficient ring* R, *the spectral sequence* $E_r(\omega_n : R)$ *degenerates at* E_2.

Proof. In any case $E_2(\omega_n : R) = E_{2n+1}(\omega_n : R)$ and the only possible nonzero differential is d^R_{2n+1}; it suffices to prove that it is zero for $R = \mathbb{Z}(\frac{1}{2})$, $\mathbb{F}_2$ and $\mathbb{Z}$.

For $R = \mathbb{Z}(\frac{1}{2})$ we see from 2.1(a) that $E_{2n+1}(\omega_n : R)$ is a free module of rank 2^{n+1} and again that $E_{2n+2}(\omega_n : R) = E_\infty(\omega_n : R)$ has rank 2^{n+1} (because $H^*(G_{n+1} : R)$ does). Thus by 1.5 we have degeneration for $R = \mathbb{Z}(\frac{1}{2})$. Using this and 1.5, $\mathrm{Im}(d^{\mathbb{Z}}_{2n+1})$ lies in the 2-torsion subgroup of $E_{2n+1}(\omega_n : \mathbb{Z})$.

To prove degeneration over $\mathbb{F}_2$, we can certainly take $n \ge 1$; then $\lambda(2n+1) = \lambda(2n+2)$ and so 2.1(b) and 1.4 yield the assertion. Change of rings gives a map of the (long-exact) Wang sequence over $\mathbb{Z}$ to the Wang sequence with $\mathbb{F}_2$-coefficients. Using the results above and an elementary diagram chase we obtain $d^{\mathbb{Z}}_{2n+1} = 0$. □

Denote by $\varepsilon = \pi^*[S^{2n+1}] \in H^{2n+1}(G_{n+1} : R)$. Again from 2.1 the restriction map

$$f^*_R : H^*(G_{n+1} : R) \to H^*(G_n : R)$$

is a surjective ring homomorphism with kernel the principal ideal (ε). Hence it is easy to see that f^*_R has an *additive* splitting and that additively we have

$$H^*(G_{n+1} : R) = H^*(G_n : R) \otimes \Lambda^*(\varepsilon)\,.$$

Once we prove that f_R^* has a multiplicative splitting, then $\varepsilon^2=0$ will give us the result.

For $R=\mathbb{F}_2$ we have proved the existence of such a splitting in Corollary 5.4.5; here we need a slight modification of it, using Corollary 5.4.4 and Proposition 5.5. In the notation of Section 5 the map

$$z_i \mapsto z_i\,, \qquad \bar{x}_s \text{ (for } G_n) \mapsto \bar{x}_s \text{ (for } G_{n+1})$$

is a ring homomorphism splitting $f_{\mathbb{F}_2}^*$. Moreover, this splitting commutes with Sq^1, although not with all the higher squaring operations in general. Now consider the universal coefficient theorem for the change of rings $p:\mathbb{Z}\to\mathbb{F}_2$ applied to G_{n+1}, G_n. Since the splitting commutes with Sq^1, 2.1(c) implies that it commutes with the 'coboundary' in the universal coefficient theorem, and hence it induces a multiplicative splitting of $f_{\mathbb{Z}}^*\otimes\mathbb{F}_2$

$$\bar{\sigma}: H^*(G_n:\mathbb{Z})\otimes\mathbb{F}_2\to H(G_{n+2}:\mathbb{Z})\otimes\mathbb{F}_2\,.$$

We will now lift $\bar{\sigma}$ to a splitting of the integral rings.

Given any space X, let Tors $H^*(X:\mathbb{Z})$ be the ideal generated by all elements of finite order in $H^*(X:\mathbb{Z})$. The quotient

$$Q(X)=H^*(X:\mathbb{Z})/\mathrm{Tors}\,H^*(X:\mathbb{Z})$$

is a functor from spaces to graded, torsion-free $\mathbb{Z}$-algebras which preserves products, as one sees from the nonfunctorial splitting in the Künneth formula. Hence for a finite (connected) H-space X, $Q(X)$ is an exterior algebra over $\mathbb{Z}$ on generators of odd degree by Hopf's theorem and Borel's Proposition 7.3 [2]. These degrees are of course well known for $X=G_n, G_{n+1}$; and thus we obtain a diagram

$$\begin{array}{ccccccccc} 0\to & \mathrm{Tors}\,H^*(G_{n+1}:\mathbb{Z}) & \to & H^*(G_{n+1}:\mathbb{Z}) & \to & \Lambda_{\mathbb{Z}}^*(y_1,\ldots,y_n,\bar{\varepsilon}) & \to 0 \\ & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} & \\ 0\to & \mathrm{Tors}\,H^*(G_n:\mathbb{Z}) & \to & H^*(G_n:\mathbb{Z}) & \to & \Lambda_{\mathbb{Z}}^*(y_1,\ldots,y_n) & \to 0 \end{array} \tag{2.3}$$

where the exact rows can be split by assigning to y_i the element x_i of (5.2.1) and to $\bar{\varepsilon}$ the element $\varepsilon=\pi^*[S^{2n+1}]$. In particular, the rows remain exact after tensoring with $\mathbb{F}_2$ and we obtain a diagram '$(2.3)\otimes\mathbb{F}_2$'.

Now again for arbitrary X, $H^*(X:\mathbb{F}_2)$ is a DGA with differential Sq^1 and from 2.1(a) and (c) we conclude that the rows of $(2.3)\otimes\mathbb{F}_2$ are precisely

$$0\to \mathrm{Sq}^1\text{-coboundaries}\to \mathrm{Sq}^1\text{-cocycles}\to \mathrm{Sq}^1\text{-cohomology}\to 0$$

for $X = G_{n+1}$ and G_n, respectively. Since σ commutes with Sq^1, it induces a splitting compatible with the diagram (2.3)$\otimes \mathbb{F}_2$.

The lifting of $\bar{\sigma}$ to σ is now already defined on the torsion-ideal by 2.1(c); put

$$\begin{array}{ccc} \mathrm{Tors}\, H^*(G_n : \mathbb{Z}) & \xrightarrow{\sigma} & \mathrm{Tors}\, H^*(G_{n+1} : \mathbb{Z}) \\ \| & & \| \\ (\mathrm{Tors}\, H^*(G_n : \mathbb{Z})) \otimes \mathbb{F}_2 & \xrightarrow{\bar{\sigma}} & (\mathrm{Tors}\, H^*(G_{n+1} : \mathbb{Z})) \otimes \mathbb{F}_2 \,. \end{array} \tag{2.4}$$

This is multiplicative and it can be extended to the middle terms of (2.3) by setting

$$\sigma(x_{i_1} \cdots x_{i_p}) = x_{i_1} \cdots x_{i_p} \,. \tag{2.5}$$

To check the multiplicativity, it is enough to note that $\sigma \otimes \mathbb{F}_2 = \bar{\sigma}$ because by (5.2.2) it is 'multiplicative' on the free summand; and $\sigma \otimes \mathbb{F}_2 = \bar{\sigma}$ follows from (2.5) and

$$\sigma(\bar{x}_i) = \bar{x}_i$$

which can be checked from (5.4.6) and our construction of $H^*(G_n : \mathbb{F}_2) \to H^*(G_{n+1} : \mathbb{F}_2)$.

This completes the proof of Theorem 1. $\square$

We record the facts which we have used from later sections:

(i) The structure of $H^*(\mathrm{Spin}(k) : \mathbb{F}_2)$ as a DGA with differential Sq^1 coming from Proposition 5.4 and Corollary 5.4.4.

(ii) The construction of the elements x_i whose square-free products generate the free part of $H^*(\mathrm{Spin}(k) : \mathbb{Z})$ (Corollary 5.3.1).

3. The integral cohomology ring of X_n and allied spaces

We will give an elementary topological calculation of the cohomology ring of

$$X_n = \mathrm{SO}(2n+1)/U(n)$$

and show that this immediately yields

(a) $H^*(\mathrm{Spin}(m)/T : R)$,

(b) the Chow rings $A^*(\mathrm{SO}(m))$, $A^*(\mathrm{Spin}(m))$,

all of which are required in later sections.

The first complete calculations of (a) and (b) are apparently in the 1974 thesis of Marlin [6] who used the intersection formulae of Chevalley and Demazure for the Bruhat cells of $\mathrm{Spin}(m)/T$ to obtain (a); then (b) follows from a well-known

result of Grothendiek (in Séminaire Chevalley 1958). Marlin's methods involve long combinatorial formulae, and they do not bring in $H^*(X_n : \mathbb{Z})$ explicitly; thus to deduce our results from his would not be much shorter than the present approach but considerably more roundabout. Similar comments apply to Toda and Watanabe [10, Section 2]. Hence we briefly set out a more direct and elementary argument below. A partial computation of $H^*(X_n : \mathbb{Z})$ by Schubert methods was recently given by Boe and Hiller [1], which further suggests that this ring has not previously been fully described. At the end of the section we compute the mod 2 cohomology of

$$X'_n = \mathrm{Spin}(2n+1)/\mathrm{SU}(n)$$

which is required for computing the action of $\mathscr{A}(2)$ on $H^*(\mathrm{Spin}(k) : \mathbb{F}_2)$, as well as the Pontryagin product in Section 6.

Fix standard inclusions $U(n) \subseteq \mathrm{SO}(2n) \subseteq \mathrm{SO}(2n+1)$ and set

$$Y_n = \mathrm{SO}(2n)/U(n)\,, \qquad X_n = \mathrm{SO}(2n+1)/U(n)\,.$$

In the associated homogeneous fibre bundle

$$\tilde{\eta}_n : Y_n \xrightarrow{i} X_n \xrightarrow{f} S^{2n}$$

the pull-back f^*TS^{2n} of the tangent bundle of S^{2n} has a tautological complex structure J. In fact, there are homogeneous complex bundles $V'_n \to X_n$, $V''_n \to Y_n$ associated to the standard $U(n)$ module $\mathbb{C}^n$; and $(f^*TS^{2n}, J) \simeq V'_n$, $i^*(V'_n) \simeq V''_n$.

Now the action of $\mathrm{SO}(2n-1) \subseteq \mathrm{SO}(2n)$ on Y_n gives an isomorphism

$$g : X_{n-1} \xrightarrow{\sim} Y_n \tag{3.1}$$

because $\mathrm{SO}(2n-1) \cap U(n) \simeq U(n-1)$ and counting dimensions, the orbital inclusion is a diffeomorphism. (In fact this isomorphism is well known in various other guises also.) The composite $j : X_{n-1} \xrightarrow{g} Y_n \xrightarrow{i} X_n$ is the standard inclusion and $g^*(V''_n) \simeq V'_{n-1} \oplus \underline{\mathbb{C}}$ as one checks easily.

Using (3.1) we can rewrite the bundle $\tilde{\eta}_n$ as

$$\eta_n : X_{n-1} \xrightarrow{j} X_n \xrightarrow{f} S^{2n}\,.$$

An obvious induction on n gives $\pi_1(X_n) = 0$, and for any R, $E_2(\eta_n : R) = E_\infty(\eta_n : R)$, $H^{\mathrm{odd}}(X_n : R) = 0$ and $H^*(X_n : R)$ is a free R-module of rank 2^n. Moreover, V'_n is stably isomorphic to V'_{n-1}.

3.2. Proposition: *There are unique elements* $\gamma_j \in H^{2j}(X_n : \mathbb{Z})$ *such that* $2\gamma_j = c_j(V'_n)$ *and* $H^*(X_n : \mathbb{Z}) = \Delta^*_{\mathbb{Z}}(\gamma_1, \ldots, \gamma_n)$.

Proof. Once we show the existence of the γ_j, the uniqueness follows because $H^*(X_n : \mathbb{Z})$ is a free $\mathbb{Z}$-module. We proceed by induction on n.

$X_1 = S^2$ and V_1' is its complex tangent bundle, so the result is true for $n = 1$. Now the degeneration of $E_*(\eta_n : \mathbb{Z})$ implies that j^* is an isomorphism in degrees $\leq 2n-1$, so for $1 \leq k \leq n-1$ we can take $\gamma_k \in H^{2k}(X_n : \mathbb{Z})$ to be the inverse image of $\gamma_k \in H^{2k}(X_{n-1} : \mathbb{Z})$. On the other hand, $V_n' \simeq (f^*(TS^{2n}), J)$ and so $c_n(V_n')$ is the Euler class of $f^*(TS^{2n}) = 2f^*[S^{2n}]$. Since $H^*(X_{n-1} : \mathbb{Z}) \simeq \Delta_{\mathbb{Z}}^*(\gamma_1, \ldots, \gamma_{n-1})$ by hypothesis and $\gamma_n^2 = f^*[S^{2n}]^2 = 0$, the rest follows. □

3.2.1. Corollary. $\gamma_1 \cdots \gamma_n = \pm[X_n]$ *in* $H^{\mathrm{top}}(X_n : \mathbb{Z})$. □

Thus we have a surjective homomorphism of graded rings

$$\varphi : \mathbb{Z}[\gamma_1, \ldots, \gamma_n] \to H^*(X_n : \mathbb{Z})$$

and to describe its kernel, the key observation is that the Pontryagin classes of V_n' are zero, because as a real, oriented bundle it is $f^*(TS^{2n})$. That is,

$$2(c_{2j} - c_1 c_{2j-1} + c_2 c_{2j-2} + \cdots) + (-1)^j c_j^2 = 0\,, \tag{3.2.2}$$

where of course $c_k = 0$ for $k > n$. The polynomials on the left-hand side of (3.2.2) are clearly in the kernel of φ; the trouble is that written out as polynomials in the γ_j they are exactly divisible by 4. Since the generators of our kernel must be indivisible, we might take the ideal generated by $\frac{1}{4} \times$ these polynomials; and this works!

3.3. Proposition. $H^*(X_n : \mathbb{Z}) = \mathbb{Z}[\gamma_1, \ldots, \gamma_n]/J_n$, *where J_n is the ideal generated by the following n homogeneous elements*:

$$\gamma_{2k} + (-1)^k \gamma_k^2 + 2\sum_{j=1}^{k-1} (-1)^j \gamma_j \gamma_{2k-j} \qquad 1 \leq k \leq n/2\,,$$

$$(-1)^l \gamma_l^2 + 2\sum_{i=2l-n}^{l-1} (-1)^i \gamma_i \gamma_{2l-i} \qquad n/2 < l \leq n\,.$$

Proof. The ideal J_n is clearly in the kernel of φ, so we have a map

$$\bar{\varphi} : \mathbb{Z}[\gamma_1, \ldots, \gamma_n]/J_n \to H^*(X_n : \mathbb{Z})$$

which is still surjective. Hence it is enough to prove that the quotient ring on the left, call it $\mathcal{H}_n$, is a free $\mathbb{Z}$-module of rank 2^n, which follows from

(a) $\dim_{\mathbb{F}_2} \mathcal{H}_n \otimes \mathbb{F}_2$ is 2^n,

(b) $\mathcal{H}_n \otimes \mathbb{Z}(\frac{1}{2})$ is a free $\mathbb{Z}(\frac{1}{2})$-module of rank 2^n.

Part (a) is immediate because mod 2 the generators of J_n simplify to $\gamma_j^2 + \gamma_{2j}$ if $j \le n/2$, γ_j^2 if $j > n/2$, and then a simple dimension count gives the dimension as 2^n.

For part (b) consider the polynomial ring $A = \mathbb{Z}[x_1, \ldots, x_n]$ in indeterminates $x_1, \ldots, x_n$ and the subring $B = \mathbb{Z}[\sigma_1, \ldots, \sigma_n]$ of all symmetric functions in $x_1, \ldots, x_n$. Let $\varphi_1, \ldots, \varphi_n$ be the elementary symmetric functions in $x_1^2, \ldots, x_n^2$. The ring

$$C_n = \mathbb{Z}[\sigma_1, \ldots, \sigma_n]/(\varphi_1, \ldots, \varphi_n)$$

is additively a free $\mathbb{Z}$-module of rank 2^n as one readily verifies[1] by identifying it with $H^*(\mathrm{Sp}(n)/U(n) : \mathbb{Z})$. There is an obvious ring homomorphism $\psi : C_n \to \mathcal{H}_n$ taking σ_j to $2\gamma_j$ and $\psi \otimes Z(\frac{1}{2})$ is clearly surjective. Since the $\mathbb{Z}$-rank of $\mathcal{H}_n$ is $\ge 2^n$, it follows that $\psi \otimes \mathbb{Z}(\frac{1}{2})$ is an isomorphism, proving (b). $\square$

Let $m = [(n+1)/2)]$ so that $2m-1$ is the largest odd integer $\le n$; and for $k \in \{1, \ldots, m\}$ define

$$a_k = \min\{2^d \mid 2^d(2k-1) > n\}.$$

3.3.1. Corollary.

$$H^*(X_n : \mathbb{F}_2) = \bigotimes_1^m \mathbb{F}_2[\bar{\gamma}_{2j-1}]/(\bar{\gamma}_{2j-1}^{a_j}), \qquad \le j \le m. \qquad \square$$

Proposition 3.3 gives the cohomology of the flag variety at one stroke. Fix a maximal torus $T' \subseteq \mathrm{SO}(m)$ and let $T \subseteq \mathrm{Spin}(m)$ be its double cover. Then

$$F(m) = \mathrm{Spin}(m)/T = \mathrm{SO}(m)/T'.$$

For $m = 2n+1$ (or $2n$) we can take $T' \subseteq U(n)$; this gives a fibre-bundle

$$U(n)/T' \to F(2n+1) \to X_n$$

and by restriction

$$U(n)/T' \to F(2n) \to Y_n$$

which are the complex flag bundles of $V_n' \to X_n$ and $V_n'' \to Y_n$ respectively. Hence by the Leray–Hirsch theorem on Chern classes we have the following:

[1] One can give a slightly longer proof by 'pure algebra'.

3.4. Proposition.

$$H^*(F(2n+1):\mathbb{Z}) = \mathbb{Z}[e_1,\ldots,e_n,\gamma_1,\ldots,\gamma_n]/(\sigma_j - 2\gamma_j, I_n)$$

and

$$H^*(F(2n):\mathbb{Z}) = \mathbb{Z}[e_1,\ldots,e_n,\gamma_1,\ldots,\gamma_n]/(\sigma_j - 2\gamma_j, \sigma_n, J_n),$$

where $e_1,\ldots,e_n$ are the Chern classes of the tautological line bundles over the flag varieties and σ_j is the jth elementary symmetric function in $e_1,\ldots,e_n$, the relation $\sigma_n = 0$ in the second ring reflecting the isomorphism $V''_n \simeq V'_{n-1} \oplus \mathbb{C}$. □

Notice that an integral basis for $H^2(F(2n+1):\mathbb{Z}) = H^2(F(2n):\mathbb{Z})$ is given by $\{\gamma_1, e_2,\ldots,e_n\}$ which corresponds to the weight lattice of T; whereas $\{e_1,\ldots,e_n\}$ only spans the lattice for T'.

3.4.1. Corollary.

$$H^*(F(2n+1):\mathbb{F}_2) \simeq H^*(U(n)/T':\mathbb{F}_2) \otimes H^*(X_n:\mathbb{F}_2)$$

and similarly for $H^*(F(2n):\mathbb{F}_2)$. □

We now introduce the so-called Chow rings of Spin(m) and SO(m). For any coefficient ring R and any compact group G with maximal torus T the differential

$$d_2^{0,1}: H^1(T:R) \to H^2(G/T:R)$$

in the R-spectral sequence of the bundle $T \to G \to G/T$ defines

$$A^*(G:R) = H^*(G/T:R)/(\text{ideal generated by } \operatorname{Im} d_2^{0,1}).$$

This is the 'Chow ring of G (with R coefficients)' and it acquires this name from the result of Grothendieck cited above. Note that since $H^*(G/T:\mathbb{Z})$ is a free $\mathbb{Z}$-module we have $A^*(G:R) = A^*(G:\mathbb{Z}) \otimes_{\mathbb{Z}} R$.

3.5. Proposition.

$$\begin{aligned} A^*(\operatorname{Spin}(2n+1):\mathbb{Z}) &= H^*(X_n:\mathbb{Z})/(\gamma_1, 2\gamma_2,\ldots,2\gamma_n) \\ &\simeq \mathbb{Z}[\bar\gamma_3,\ldots,\bar\gamma_{2j-1},\ldots,\bar\gamma_{2m-1}]/(\bar\gamma_{2j-1}^{a_j}) \end{aligned}$$

where a_j, m are as in Corollary 3.3.1.

Proof. The second description follows from the first as in 3.3.1, because in

positive degrees $A^*(\mathrm{Spin}(2n+1):\mathbb{Z})$ is an $\mathbb{F}_2$-vector space by the first description. Now the composite

$$H^*(X_n:\mathbb{Z}) \to H^*(F(2n+1):\mathbb{Z}) \to A^*(\mathrm{Spin}(2n+1:\mathbb{Z}))$$

is surjective by Proposition 3.4 and the fact that $\mathrm{Spin}(2n+1)$ is 2-connected, so that the ideal generated by $\operatorname{Im} d_2^{0,1}$ is $(\gamma_1, e_2, \ldots, e_n)$. Under the usual operation of the Weyl group W on $H^*(F(2n+1):\mathbb{Z})$, $H^*(X_n:\mathbb{Z})$ is precisely the subring invariant under the symmetric group $S_n \subseteq W$, as we see from Proposition 3.4. Hence

$$H^*(X_n:\mathbb{Z}) \cap (\gamma_1, e_2, \ldots, e_n) = (\gamma_1, \sigma_2, \ldots, \sigma_n) = (\gamma_1, 2\gamma_2, \ldots, 2\gamma_n)$$

giving the desired assertion. □

3.5.1. Corollary.

$$\begin{aligned} A^*(\mathrm{Spin}(2n+1):\mathbb{F}_2) &\simeq H^*(X_n:\mathbb{F}_2)/(\gamma_1) \\ &\simeq \mathbb{F}_2[\bar{\gamma}_3, \ldots, \bar{\gamma}_{2m-1}]/(\gamma_{2j-1}^{2j}). \quad \square \end{aligned}$$

In the same way we can calculate $A^*(\mathrm{Spin}(2n))$ and $A^*(\mathrm{SO}(m))$. We state the results without proof.

3.6. Proposition.

(a) $A^*(\mathrm{Spin}(2n):\mathbb{Z}) = H^*(X_{n-1}:\mathbb{Z})/(\gamma_1, 2\gamma_2, \ldots, 2\gamma_{n-1})$,

(b) $A^*(\mathrm{SO}(2n+1):\mathbb{Z}) = H^*(X_n:\mathbb{Z})/(2\gamma_1, \ldots, 2\gamma_n)$,

(c) $A^*(\mathrm{SO}(2n):\mathbb{Z}) = H^*(X_{n-1}:\mathbb{Z})/(2\gamma_1, \ldots, 2\gamma_{n-1})$. □

Notice that $A^+(\mathrm{SO}(2n+1):\mathbb{Z}) \simeq H^+(X_n:\mathbb{F}_2)$ and similarly for $A^+(\mathrm{SO}(2n):\mathbb{Z})$; that is, except for the factor $\mathbb{Z}$ in degree 0, the integral Chow ring for $\mathrm{SO}(2n+1)$ (respectively $\mathrm{SO}(2n)$) is the same as the mod 2 cohomology of X_n (respectively X_{n-1}). As we shall need the Chow rings of $\mathrm{Spin}(2n+1)$ and $\mathrm{SO}(2n+1)$ in the subsequent sections, we use the abbreviations $A_n^*(R) = A^*(\mathrm{Spin}(2n+1):R)$ and $A_n'^*(R) = A^*(\mathrm{SO}(2n+1):R)$.

Finally, we make some remarks on the cohomology of $X_n' = \mathrm{Spin}(2n+1)/\mathrm{SU}(n)$ which arises from the lift of $\mathrm{SU}(n) \to U(n) \to \mathrm{SO}(2n+1)$ to $\mathrm{Spin}(2n+1)$. The principal circle-bundle

$$S^1 \to X_n' \to X_n$$

has Euler class γ_1 because X_n' is 3-connected. Define

$$I_n^*(R) = \{ y \in H^*(X_n : R) \mid \gamma_1 \cdot y = 0 \} .$$

Then the Gysin sequence of the S^1-bundle above gives

$$\begin{aligned} &H^{2k+1}(X_n' : R) \simeq I_n^{2k}(R) , \\ &H^{\mathrm{ev}}(X_n' : R) = H^*(X_n : R)/(\gamma_1) , \end{aligned} \tag{3.6.1}$$

because $H^{\mathrm{odd}}(X_n : R) = 0$. For $R = \mathbb{F}_2$ we obtain the following more explicit description from Corollary 3.3.1.

3.7. Proposition. *There is a unique element* $x \in H^{2a_1-1}(X_n' : \mathbb{F}_2)$ *such that* $H^*(X_n' : \mathbb{F}_2)$ *is a free module over* $A_n^*(\mathbb{F}_2)$ *on* 1 *and* x.

Proof. From (3.6.1) and Corollary 3.3.1 we see that

$$\begin{aligned} &I_n^*(\mathbb{F}_2) = (\bar{\gamma}_1^{a_1-1}) A_n^*(\mathbb{F}_2) , \\ &A_n^*(\mathbb{F}_2) = H^*(X_n : \mathbb{F}_2)/(\bar{\gamma}_1) \end{aligned}$$

and then the result follows by defining $x \in H^{2a_1-1}(X_n' : \mathbb{F}_2)$ to be the element which maps to $\bar{\gamma}_1^{a_1-1} \in I_n^*(\mathbb{F}_2)$. □

The ring structure of $H^*(X_n' : \mathbb{F}_2)$ will thus be determined by specifying $x^2 \in A_n^*(\mathbb{F}_2)$. In fact, we shall describe $\mathrm{Sq}^i(x)$ for all i, which are needed for the $\mathcal{A}(2)$-module $H^*(\mathrm{Spin}(2n+1) : \mathbb{F}_2)$ (Section 5). $\mathrm{Sq}^i(x)$ for $i \geq 2$ is given below; $\mathrm{Sq}^1(x)$ will be calculated in Lemma 5.4.3 by comparing x with the reduction of an integral class which comes from the element ψ_s of Proposition 4.5.

3.8. Proposition. $\mathrm{Sq}^i(x) = 0$ *for* $i \geq 2$.

Proof. First suppose i is even; then $\mathrm{Sq}^i(x) \in H^{\mathrm{odd}}(X_n' : \mathbb{F}_2)$ which injects under the Gysin coboundary into $H^*(X_n : \mathbb{F}_2)$. Using the Thom isomorphism and the commutation of Sq^j with coboundaries it is enough to check that $\mathrm{Sq}^j(\bar{\gamma}_1^{a_1-1}) = 0$ for $j \geq 1$, which follows from the Cartan formula. If $i = 2j + 1 \geq 3$, then the Adem relation $\mathrm{Sq}^{2j+1} = \mathrm{Sq}^1\mathrm{Sq}^{2j}$ reduces the assertion to the previous case. □

In particular, we have $x^2 = 0$.

3.8.1. Corollary. $H^*(X_n' : \mathbb{F}_2) \simeq A_n^*(\mathbb{F}_2) \otimes \Lambda_{\mathbb{F}_2}^*(x)$. □

We end with some miscellaneous remarks.

3.9.1. For $1 \leq n_1 < n_2$ the inclusion $X_{n_1} \to X_{n_2}$ is an isomorphism on $H^2(? : R)$,

hence there is also an inclusion of principal circle-bundles, $X'_{n_1} \to X'_{n_2}$. If there is no power of 2 between n_1 and n_2, then the height of γ_1 in $H^*(X_{n_1} : \mathbb{F}_2)$ is the same as in $H^*(X_{n_2} : \mathbb{F}_2)$ so the element x for X'_{n_2} restricts to x for X'_{n_1}.

3.9.2. The isomorphism $X_{n-1} \simeq Y_n$ of (3.1) lifts to an isomorphism of the corresponding circle-bundles

$$X'_{n-1} \simeq Y'_n = \mathrm{Spin}(2n)/\mathrm{SU}(n)$$

and hence the class x for Y'_n is the same as that for X'_{n-1}. This will be used in comparing the mod 2 cohomology rings of Spin$(2n-1)$ and Spin$(2n)$ (see Propositions 5.4 and 6.2).

3.9.3. For $n \geq 5$, x is not the reduction of a class in $H^{2a_1-1}(X'_n : \mathbb{Z})$. This can be seen as follows. From $H^{\mathrm{odd}}(X'_n : \mathbb{Z}) \simeq I^*_n(\mathbb{Z})$ (with a shift) and the Hard Lefschetz theorem applied to the Hodge manifold X_n we obtain that $I^*_n(\mathbb{Q}) = 0$ below the middle dimension. Hence the coboundary $H^{2k-1}(X'_n : \mathbb{F}_2) \to \mathrm{Tor}(H^{2k}(X'_n : \mathbb{Z}), \mathbb{F}_2)$ is an isomorphism for $2k - 1 \leq (n^2 + n)/2$. Our formula for $\mathrm{Sq}^1(x)$ together with Corollary 3.8.1 will imply the rather subtle fact that the 2-torsion subgroup of $H^{2a_1}(X'_n : \mathbb{Z})$ is cyclic of order 2 if $n \neq 2^a$, and of order ≥ 4 otherwise.

4. The E_3-term of the spectral sequence for ξ_n

We will describe the bigraded algebra $E_3^{*,*}(\xi_n : \mathbb{Z})$ for

$$\xi_n : T \to \mathrm{Spin}(2n+1) \to F(2n+1) = \mathrm{Spin}(2n+1)/T$$

and use the description to show that $E_3(\xi_n : \mathbb{Z}) = E_\infty(\xi_n : \mathbb{Z})$. In the next section we will 'disentangle' the filtration to compute $H^*(\mathrm{Spin}(2n+1) : \mathbb{Z})$ from $E_\infty(\xi_n : \mathbb{Z})$. The corresponding calculations in the even case are quite similar and will be briefly indicated from time to time as necessary.

Of course, the algebras $E_3(\xi_n : \mathbb{Z}(\frac{1}{2}))$ and $E_3(\xi_n : \mathbb{F}_2)$ are much simpler to calculate, and will be done first. They will then be compared with $E_3(\xi_n : \mathbb{Z})$ by means of the following 'universal coefficient formula' for $E_3(\xi_n)$. For any compact connected lie group G with maximal torus T, and any coefficient ring R, the differential d_2^R in E_2 of

$$T \to G \to G/T$$

is 'universally' specified, so that $d_2^R = d_2^{\mathbb{Z}} \otimes R$. Since $E_2(R)$ is free we obtain

$$0 \to E_3(G : \mathbb{Z}) \otimes R \to E_3(G : R) \xrightarrow{\delta} \mathrm{Tor}(E_3(G : \mathbb{Z}), R) \to 0$$

where δ has bidegree $(2, -1)$. Using the isomorphism $d_1^{0,1}: H^1(T:\mathbb{Z}) \to H^2(F(2n+1):\mathbb{Z})$ (Spin$(2n+1)$ is 2-connected) we fix for once and for all, elements $t, t_1, \ldots, t_n \in H^1(T:\mathbb{Z})$ such that

$$d_2(1 \otimes t) = \gamma_1 \otimes 1\,, \qquad d_2(1 \otimes t_i) = e_i \otimes 1\,, \quad 1 \le i \le n\,.$$

Then $\{t, t_2, \ldots, t_n\}$ is a $\mathbb{Z}$-basis for $H^1(T:\mathbb{Z})$, $\{t_1, \ldots, t_n\}$ spans the image $H^1(T':\mathbb{Z}) \to H^1(T:\mathbb{Z})$ ($T' \subseteq \mathrm{SO}(2n+1)$) and

$$2t = \sum_1^n t_i\,.$$

4.1. Proposition. *If* 2 *is invertible in* R, *then*

$$E_3(\xi_n : R) = \Lambda_R^*(\psi_1, \ldots, \psi_n)\,, \qquad \text{bidegree}(\psi_j) = (4j-2, 1)$$

and hence

$$E_3(\xi_n : R) = E_\infty(\xi_n : R) = H^*(\mathrm{Spin}(2n+1) : R)\,.$$

Proof. This is of course known, but the same proof is required again, and is short enough to give.

Combining Corollary 3.3.1 and Proposition 3.4 with the hypothesis that $\frac{1}{2} \in R$,

$$H^*(F(2n+1):R) = R[e_1, \ldots, e_n]/(\varphi_1, \ldots, \varphi_n)$$

where $\varphi_j = j$th elementary symmetric function in $e_1^2, \ldots, e_n^2$. Hence the cochain complex $(E_2(\xi_n : R), d_2^R)$ is a Koszul complex whose defining ideal of relations is generated by the regular sequence $\{\varphi_1, \ldots, \varphi_n\}$. Therefore (see [9]), the cohomology is an exterior algebra on n generators of degree $(\deg \varphi_j) - 1 = 4j - 1$. In fact, Tate's construction [9] implies that the fibre-degree of each generator is 1, which gives $E_3 = E_\infty$ and then since char $R \neq 2$, the last isomorphism. □

An entirely similar argument works for Spin$(2n)$.

4.1.1. Corollary. *The torsion ideal of* $E_3(\xi_n : \mathbb{Z})$ *has order a power of* 2, *and the* $\mathbb{Z}$-*rank of* $E_3(\xi_n : \mathbb{Z})$ *is* 2^n. □

We will now construct d_2-cocycles $\psi_k \in E_2^{4k-2,1}(\xi_n : \mathbb{Z})$ which generate an exterior subalgebra of $E_2(\xi_n : \mathbb{Z})$ mapping isomorphically onto the *free part* of $E_3(\xi_n : \mathbb{Z})$. This is one of the main tools of this paper. To give the formulae we need some algebraic preliminaries.

Let $\sigma_1, \ldots, \sigma_n$ be the elementary symmetric functions in $e_1, \ldots, e_n$ and s

$$\sigma_k(i) = k\text{th elementary symmetric function in } \{e_{i+1}, \ldots, e_n\}$$

so that $\sigma_k(i) = 0$ if $k > n - i$. Then we have

$$\sigma_k(i) = \sigma_k(i+1) + \sigma_{k-1}(i+1)e_{i+1} \tag{4.1.2}$$

and these identities have the following consequences. In the (acyclic) Koszul complex

$$K = \Lambda^*_{\mathbb{Z}}(t_1, \ldots, t_n) \otimes \mathbb{Z}[e_1, \ldots, e_n],$$
$$d(t_i \otimes 1) = 1 \otimes e_i, \qquad d(1 \otimes e_j) = 0$$

define the elements

$$\tau_k = \sum_{j=1}^{n} t_j \otimes \sigma_k(j) \quad 1 \le k \le n .$$

Then

$$\begin{aligned} &\text{(a)} \qquad d\tau_k = 1 \otimes \sigma_k \quad \text{in } K^*, \\ &\text{(b)} \qquad \tau_1 \cdots \tau_n = \pm t_1 \cdots t_n \otimes e_2 e_3^2 \cdots e_n^{n-1} . \end{aligned} \tag{4.1.3}$$

These formulae persist in suitable quotients of K^* and they imply the following proposition for the bundle

$$\nu_n : T' \to U(n) \to U(n)/T' ,$$

where $T' \subseteq U(n)$ is a maximal torus:

4.2. Proposition. *$\tau_1, \ldots, \tau_n$ are d_2-cocycles in $E_2(\nu_n : \mathbb{Z})$ and generate the exterior algebra $E_3(\nu_n : \mathbb{Z}) = E_\infty(\nu_n : \mathbb{Z})$.*

Proof. Let $H^*(U(n)/T : \mathbb{Z}) = \mathbb{Z}[e_1, \ldots, e_n]/(\sigma_1, \ldots \sigma_n)$. It is well known that $e_2 e_3^2 \cdots e_n^{n-1} = [U(n)/T]$. Then (4.1.3(b)) together with 1.6 implies that the exterior algebra generated by $\{\tau_1, \ldots, \tau_n\}$ is a summand of $E_3(\nu_n : \mathbb{Z})$. On the other hand, $E_3(\nu_n : \mathbb{Z})$ is a free rank 2^n—in fact, exterior—by the analogue of Proposition 4.1. Thus $E_3(\nu_n : \mathbb{Z})$ is generated by elements of fibre-degree 1 which gives the collapsing. □

Note that $E_2^{p,q}(\nu_n : \mathbb{Z}) = 0$ if p is odd, and the same is therefore true of $E_\infty^{p,q}(\nu_n : \mathbb{Z})$. This implies that (see Lemma 5.2) $E_\infty^{2k,1}(\nu_n : \mathbb{Z}) \hookrightarrow H^{2k+1}(U(n) : \mathbb{Z})$ and under this injection the elements τ_j go into elements which transgress to $\pm c_j$ (the universal Chern classes).

Returning to Spin$(2n+1)$, we wish to construct elements $\theta_1, \ldots, \theta_n \in E_2(\xi_n : \mathbb{Z})$ in analogy with the τ_j above, whose d_2-images are the generating polynomials of the ideal J_n. We start out with something slightly weaker; namely, by constructing θ_i so that

$$d_2(\theta_i) = 2(i\text{th generator of } J_n)\,.$$

That is, multiplying the generators of J_n by 2, we obtain

$$\sigma_{2j} + (-1)^j \sigma_j \gamma_j + \sum_{i=1}^{j-1} (-1)^i \sigma_i \sigma_{2j-i}$$

with $\sigma_k = 0$ if $k > n$. The elements τ_j can be defined in $E_2(\xi_n : \mathbb{Z})$ also; hence letting

$$\begin{aligned} \theta_j &= \tau_{2j} + (-1)^j \tau_j \gamma_j + \sum_{i=1}^{s-1} (-1)^i \sigma_i \tau_{2j-1} \quad & i \le j \le n/2\,, \\ \theta_j &= (-1)^j \tau_j \gamma_j + \sum_{i=2j-n}^{j-1} (-1)^i \sigma_i \tau_{2j-1} \quad & n/2 < j \le n\,, \end{aligned} \tag{4.2.1}$$

we have the following proposition:

4.3. Proposition. (a) $d_2\theta_j = 0$ *in* $E_2(\xi_n : \mathbb{Z})$.
(b) $\theta_j\theta_k = -\theta_k\theta_j$ *and so* $\theta_k^2 = 0$ *in* $E_2(\xi_n : \mathbb{Z})$.
(c) $\theta_1 \wedge \cdots \wedge \theta_n = \pm(\tau_1 \wedge \cdots \wedge \tau_n) \otimes \gamma_1 \cdots \gamma_n$.

Proof. (a) follows from construction and (b) from the fact that the total degree of θ_j is $4j-1$. For (c) we first remark that in $H^*(X_n : \mathbb{Z})$ we have the identity

$$\gamma_k^2 \gamma_{k+1} \cdots \gamma_n = 0 \tag{4.3.1}$$

which is proved by descending induction on k, starting with $\gamma_n^2 = 0$ and noting that γ_k^2 is in the ideal $(\gamma_{k+1}, \ldots, \gamma_n)$ in $H^*(X_n : \mathbb{Z})$. Now we calculate θ_n, $\theta_n \wedge \theta_{n-1}, \ldots$, etc. inductively. If $k > n/2$, then the leading term of θ_k is $\pm\tau_k\gamma_k$ and one easily checks from (4.3.1) that it is the only one which contributes to the product $\theta_n \cdots \theta_k$. If $k \le n/2$,

$$\theta_k = \tau_{2k} \pm \tau_k \gamma_k \mod(\gamma_{k+1}, \ldots, \gamma_n)$$

and since τ_{2k} and $\gamma_{k+1}, \ldots, \gamma_n$ already occur in the product $\theta_n \wedge \cdots \wedge \theta_{k+1}$, the only term which contributes to the wedge product is $\pm\tau_k\gamma_k$. □

Using Corollary 3.2.1 and (4.1.3) we obtain the following corollary:

4.3.2. Corollary. $\theta_1 \wedge \cdots \wedge \theta_n = \pm t_1 \wedge \cdots \wedge t_n \otimes [F(2n+1)]$. $\square$

Now

$$t_1 \wedge \cdots \wedge t_n = 2t \wedge t_2 \wedge \cdots \wedge t_n = 2[T] \in H^n(T : \mathbb{Z})$$

so to obtain $[T] \otimes [F(2n+1)]$ over $\mathbb{Z}$ we must modify the θ_j. Since

$$\tau_1 = \sum_1^n t_j \otimes 1 = 2t \otimes 1 \quad \text{in } E_2(\xi_n : \mathbb{Z}) ,$$

we have

$$\theta_1 = \tau_2 - 2t \otimes 1 .$$

We will now successively modify the elements θ_j, $j = 2^k \leq n$ so that the *last* one θ'_s $(s = 2^\alpha \leq n < 2^{\alpha+1})$ is divisible by 2. Put

$$\theta'_1 = \theta_1 , \qquad \theta'_{2^i} = \theta_{2^i} - \theta'_{2^{i-1}}(\gamma_{2^i}) \quad 1 \leq i \leq \alpha$$

and for $j \neq 2^i$, $\theta'_j = \theta_j$.

4.4. Proposition. *$\theta'_1 \wedge \cdots \wedge \theta'_n = \theta_1 \wedge \cdots \wedge \theta_n$ and θ'_s is divisible by* 2.

Proof. The first assertion is clear and the second follows from

$$\theta_1 \equiv \tau_2 \mod 2 , \qquad \theta_j \equiv \tau_{2j} + \tau_j \gamma_j \mod 2 \quad \textit{for } 2 \leq j \leq n/2 ,$$
$$\theta_j \equiv \tau_j \gamma_j \mod 2 \quad \textit{for } n/2 < j \leq n . \qquad \square$$

Define elements $\psi_1, \ldots, \psi_n \in E_2(\xi_n : \mathbb{Z})$ by

$$\begin{aligned} &\psi_j = \theta'_j = \theta_j && \text{if } j \neq 2^i , \\ &\psi_j = \theta'_j && \text{if } j = 2^i < 2^\alpha = s , \\ &\psi_s = \tfrac{1}{2}\theta_s . \end{aligned} \tag{4.4.1}$$

Then the ψ_i are clearly d_2-closed and by Propositions 4.3 and 4.4,

$$\psi_1 \wedge \cdots \wedge \psi_n = t \cdots t_n \otimes [F(2n+1)] = [T] \otimes [F(2n+1)] .$$

Let $\mathscr{E}^*_n$ be the subalgebra of $E_2(\xi_n : \mathbb{Z})$ generated by $\{1, \psi_1, \ldots, \psi_n\}$.

4.5. Proposition. *$\mathscr{E}^*_n$ is an exterior algebra on $\psi_1, \ldots, \psi_n$ which is an additive*

summand in $E_2(\xi_n : \mathbb{Z})$ *and which injects into* $E_3(\xi_n : \mathbb{Z})$. *In fact*, $\mathscr{E}_n^* =$ *free part of* $E_3(\xi_n : \mathbb{Z})$.

Proof. The last statement follows from Corollary 4.1.1 and the first sentence of the proposition. The claims in the first sentence are all of the following type: using an obvious multi-index notation, 'if

$$\sum a_I \psi_I = 0 \text{ (mod } p \text{ or mod } d_2(E_2)),$$

then each $a_I = 0$ (mod p or over $\mathbb{Z}$)'. They are all proved in the same way. If some $a_I \neq 0$ (mod p or over $\mathbb{Z}$), then choosing the complementary index set I' and multiplying the above equation by $\psi_{I'}$, one gets

$$a_I[T] \otimes [F(2n+1)] = 0 \text{ (mod } p \text{ or mod } d_2(E_2)),$$

which is impossible.

Now $\mathscr{E}_n^*$ is contained in the d_2-cocycles, and by the argument above intersects the d_2-coboundaries in 0, hence it injects into $E_3(\xi_n : \mathbb{Z})$ and since it is a retract of $E_2(\xi_n : \mathbb{Z})$ it is a summand of $E_3(\xi_n : \mathbb{Z})$, of rank 2^n. □

For Spin$(2n+2)$, the corresponding assertion holds with $\psi_1, \ldots, \psi_n$ (given by the same formulae) and τ_{n+1} as generators (see Propositions 4.2 and 3.5).

The following result is given in [4]; but we need a more explicit version which gives cocycle representatives for the generators.

4.6. Proposition.

$$E_3^{*,*}(\xi_n : \mathbb{F}_2) = \Lambda^*(\rho_2, \ldots, \rho_n) \otimes \Lambda(t \otimes \bar{\gamma}_1^{a_1 - 1}) \otimes A_n^*(\mathbb{F}_2)$$

where bidegree $\rho_j = (2j-2, 1)$. *Since* $E_3^{*,*}(\xi_n : \mathbb{F}_2)$ *is generated by elements of fibre degree* ≤ 1, $d_r = 0$ *for* $r \geq 3$ *and* $E_3(\xi_n : \mathbb{F}_2) = E_\infty(\xi_n : \mathbb{F}_2)$.

Proof. From Corollary 3.4.1, using Corollaries 3.3.1 and 3.5.1 we see that $E_2(\xi_n : \mathbb{F}_2)$ decomposes as the tensor-product of three (bigraded) cochain complexes:

$$A_n^*(\mathbb{F}_2) \quad \text{with } d_2 = 0,$$
$$B_n^* = \Lambda^*(t) \otimes \mathbb{F}_2[\bar{\gamma}_1]/(\bar{\gamma}^{a_1}) \quad \text{with } d_2(t) = \bar{\gamma}_1,$$
$$C_n^* = \Lambda^*(t_2, \ldots, t_n) \otimes H^*(U(n)/T' : \mathbb{F}_2) \quad \text{with } d_2(t_i) = e_i.$$

The cohomology algebras of the first two complexes give the last two factors in the assertion. The cohomology of C_n^* is exterior on $n-1$ generators by the

argument of Proposition 4.1 and cocycle representatives in E_2 are given by the formulae

$$\rho_j = \sum_2^n t_i \otimes (\sigma_{j-1}(i) - \sigma_{j-1}(1)) .$$

They satisfy the analogue of Proposition 4.2; indeed, for $2 \leq j \leq n$

$$\tau_j - \rho_j = \left(\sum_1^n t_i\right) \otimes \sigma_{j-1}(1) = 2t \otimes \sigma_{j-1}(1)$$

so that (4.1.3(b)) proves the claim. □

A similar computation works for the $\mathbb{F}_2$ spectral sequence of

$$T \to \mathrm{Spin}(2n+2) \to F(2n+2)$$

with an extra exterior factor $\Lambda^*(\tau_{n+1})$ tagging along from E_2 onwards. The appearance of the cohomology of $\mathrm{SU}(n)$ in the proof above is not just an algebraic happenstance. The mod 2 spectral sequence of $\mathrm{SU}(n) \to \mathrm{Spin}(2n+1) \to X_n'$ degenerates at E_2 (as is easily checked) and we will use this fact in Section 5. The equations

$$\tau_j = \rho_j \mod 2 \quad 2 \leq j \leq n$$

enable one to write down the mod 2 reductions $\bar{\psi}_k$. First

$$\begin{aligned} \bar{\theta}_1 &= \rho_2 , \\ \bar{\theta}_k &= \rho_{2k} + \rho_k \gamma_k \quad 2 \leq k \leq n/2 , \\ \bar{\theta}_k &= \rho_k \gamma_k \qquad\quad n/2 < k \leq n . \end{aligned} \tag{4.6.1}$$

Hence if $j \neq 2^i$,

$$\begin{aligned} \bar{\psi}_j &= \rho_{2j} + \rho_j \gamma_j \quad 2 \leq j \leq n/2 , \\ \bar{\psi}_j &= \rho_j \gamma_j \qquad\quad n/2 \leq j \leq n , \end{aligned} \tag{4.6.2}$$

while if $j = 2^i < 2^\alpha = s$,

$$\bar{\psi}_j = \rho_{2j} . \tag{4.6.3}$$

The mod 2 reduction of ψ_s takes some further work as its definition involves a division by 2. So we work mod 4 and then 'pull-back' the result under the

injection $\mathbb{F}_2 \to \mathbb{Z}/4$. First of all, $\tau_1\sigma_j = 0 \bmod 4$ because $\tau_1 = 2t$. Hence in the expression for θ_j ($j = 2^i$), we can drop the τ_1-term. Secondly, an elementary induction shows that since

$$\theta_s = \tau_s\gamma_s + \sum_{2s-n}^{s-1} (-1)^i \tau_i \sigma_{2s-1}$$

and each of the σ_{2s-i} has an index $>s$

$$\psi_s(\xi_n) = `\psi_s(\xi_s)' + \sum_{2s-n}^{s-1} (-1)^i \tau_i \sigma_{2s-1}$$

where the expression in quotes is formally the same as we would obtain in $E_2(\xi_s : \mathbb{Z}/4)$. So it is enough to do the calculation of $\psi_s(\xi_s)$.

Again elementary inductive computations (induction on 2^i) give

(a) $\psi_s(\xi_s)$ has 'leading-term' $\pm t \otimes \gamma_1\gamma_2 \cdots \gamma_s$,

(b) the difference

$$\psi_s(\xi_s) \mp t \otimes \gamma_1 \cdots \gamma_s = (?)\gamma_s, \tag{4.6.4}$$

where the expression in parentheses will become d_2-closed in $E_2(\xi_s : \mathbb{F}_2)$, because $d_2\tau_j = \sigma_j = 0 \bmod 2$. Now working in $\mathbb{F}_2$,

$$\bar{\gamma}_{2^i} = \bar{\gamma}_1^{2^i}$$

so

$$\bar{\psi}_s(\xi_s) = t \otimes \bar{\gamma}_1^{a_1-1} + (?)\bar{\gamma}_1^s \quad (a_1 = 2^\alpha)$$

and therefore the last summand is exact mod 2:

$$d_2(t \otimes (?)\bar{\gamma}_1^{s-1}) = (?)\bar{\gamma}_1^s .$$

Thus modulo exact terms in $E_2(\xi : \mathbb{F}_2)$ we have

$$\begin{aligned} \bar{\psi}_s(\xi_s) &\equiv t \otimes \bar{\gamma}_1^{a_1-1}, \\ \bar{\psi}_s &\equiv t \otimes \bar{\gamma}_1^{a_1-1} + \sum_{2s-n}^{s-1} \rho_j \gamma_{2s-j} . \end{aligned} \tag{4.6.5}$$

Note that for $n \leq 8$ there is no second term in (4.6.5); and for any n we can use $\bar{\psi}_s$ instead of $t \otimes \bar{\gamma}_1^{a_1-1}$ as a generator of $E_3(\xi_n : \mathbb{F}_2)$.

Using Proposition 4.6 and the universal-coefficient formula from the beginning of the section,

$$0 \to E_3(\xi_n : \mathbb{Z}) \otimes \mathbb{F}_2 \to E_3(\xi_n : \mathbb{F}_2) \xrightarrow{\delta} \mathrm{Tor}(E_3(\xi_n : \mathbb{Z}) : \mathbb{F}_2) \to 0 ,$$

we can analyze the torsion ideal $T_n^{*,*}$ = torsion elements of $E_3^{*,*}(\xi_n : \mathbb{Z})$. First consider the rank of T_n. For any finitely-generated abelian group Γ, put

$$r_0(\Gamma) = \dim_Q(\Gamma \otimes \mathbb{Q}), \qquad r_p(\Gamma) = \dim_{\mathbb{F}_p}(\Gamma \otimes \mathbb{F}_p) - r_0(\Gamma).$$

Define α_j by

$$a_j = 2^{\alpha_j},$$

a_j as in Section 3.

4.7. Proposition. *$r_p(T_n) = 0$ for $p > 2$, $r_2(T_n) = 2^{n-1}(2^{n-\alpha_1} - 1)$, so that $T_n = 0$ for $n = 1, 2$.*

Proof. The first assertion is simply that $E_3(\xi_n : \mathbb{Z}) \otimes \mathbb{Z}(\frac{1}{2})$ is free and the second follows from Propositions 4.1 and 4.6. $\square$

Next, we prove that T_n is a group of exponent 2 by using a 'formal' Bockstein argument. In the universal-coefficient exact sequence above, if we follow the map δ by reduction mod 2, and show that the composite $\bar{\delta}$ satisfies

$$\dim_{\mathbb{F}_2}(\operatorname{Im} \bar{\delta}) = r_2(T_n),$$

then we will have proved that $2T_n = 0$.

From the interpretation of $\bar{\delta}$ as a coboundary, it follows that

$$\bar{\delta} : E_3^{*,*}(\xi_n : \mathbb{F}_2) \to E_3^{*,*}(\xi_n : \mathbb{F}_2)$$

is a derivation of a bidegree $(2, -1)$ and $\bar{\delta}^2 = 0$. Moreover, $\bar{\delta}$ is given on the generators by

(a) $\bar{\delta}(\gamma_j) = 0$ because γ_j has fibre degree 0.

(b) If $k = 2^i$, $0 \le i < \alpha$, $\bar{\delta}(\rho_{2k}) = \bar{\delta}(\psi_k) = 0$ because ψ_k is the reduction of an integral class.

(c) Similarly, $\bar{\delta}(\bar{\psi}_s) = 0$.

(d) If $k \ne 2^i$, $\bar{\delta}(\rho_k) = \bar{\gamma}_k$ since τ_k is an integral cochain mapping to ρ_k. For computational purposes we rewrite (d) as:

(d′) Put $k = 2^b(2j-1)$, $j \ge 2$. Then

$$\bar{\delta}(\rho_k) = \bar{\gamma}_{2j-1}^{2^b}.$$

4.8. Proposition. $\dim_{\mathbb{F}_2}(\operatorname{Im} \bar{\delta}) = 2^{n-\alpha}(2^{n-\alpha_1} - 1)$, *so* $2 \cdot T_n = 0$.

Proof. We know $\dim_{\mathbb{F}_2}(E_3(\xi_n : \mathbb{F}_2))$ from Proposition 4.6, so it is equivalent to

prove that

$$\dim_{\mathbb{F}_2} H(E_3(\mathbb{F}_2), \bar{\delta}) = 2^n .$$

For $i \leq j \leq m = [(n+1)/2]$ define subalgebras of $E_3(\xi_n : \mathbb{F}_2)$ as follows.

$$S_1 = \Lambda^*(\rho_2, \rho_4, \ldots, \rho_{2^{\alpha_1 - 1}})$$

and for $2 \leq j \leq m$

$$S_j = \Lambda^*(\rho_{(2j-1)}, \rho_{2(2j-1)}, \ldots, \rho_{2^{\alpha_j - 1}(2j-1)}) \otimes \mathbb{F}_2[\bar{\gamma}_{2j-1}]/(\bar{\gamma}_{2j-1}^{a_j})$$

and note that (using (4.6.5))

$$E_3(\xi_n : \mathbb{F}_2) = \bigotimes_{j=1}^{m} S_j \otimes \Lambda^*(\bar{\psi}_s) , \qquad \bar{\delta}(S_j) \subseteq S_j , \qquad \bar{\delta}(\bar{\psi}_s) = 0$$

whence

$$H(E_3, \bar{\delta}) = \bigotimes_{j=1}^{m} H(S_j, \bar{\delta}) \otimes \Lambda^*(\bar{\psi}_s)$$

and then our claim amounts to the following:

4.8.1. Lemma. $\dim H(S_j, \bar{\delta}) = 2^{\alpha_j}$, $1 \leq j \leq m$.

Proof. From the explicit formulae for $\bar{\delta}$ on the generators, (b) gives $H(S_1, \bar{\delta}) = S_1$ which has the required dimension. The remaining assertions are all special cases of the following paradigm example which will play a basic role hereafter. Define a DGA

$$S = \Lambda^*_{\mathbb{F}_2}(y_0, \ldots, y_{t-1}) \otimes \mathbb{F}_2[x]/(x^{2^t}) ,$$

where the degrees of the elements are so arranged that the map $\partial : S \to S$ given by

$$\partial(y_i) = x^{2^i} , \qquad \partial(x) = 0$$

and extended to S as a derivation has degree 1. Then the assertions of the lemma amount to the statement that $\dim_{\mathbb{F}_2} H(S, \partial) = 2^t$. There is a ∂-stable filtration of S by ideals

$$F^r S = \Lambda^*(y_0, \ldots, y_{t-1}) \otimes (x^{2^r}) , \quad 0 \leq r \leq t-1 .$$

We will show that $\dim_{\mathbb{F}_2} H(S/F^r S, \partial) = 2^t$, independent of r. We first make an

explicit calculation to show $\dim_{\mathbb{F}_2} H(S/F^rS, \partial) \geq 2^t$. Define

$$\eta_j = y_{j+1} - y_j \otimes x^{2^j}, \quad 0 \leq j \leq r-1 .$$

These elements, together with y_i, $r \leq i \leq t-1$ are closed in S/F^rS and of square 0; and

$$\eta_0 \wedge \cdots \wedge \eta_{r-1} \wedge y_r \wedge \cdots \wedge y_{t-1} = y_0 \wedge \cdots \wedge y_{t-1} \otimes x^{2^r-1}$$

which is the 'fundamental class' in S/F^rS, whence by the argument of 1.6, the exterior algebra generated by $\eta_0, \ldots, \eta_{r-1}, y_r, \ldots, y_{t-1}$ injects into $H(S/F^rS, \partial)$, giving the lower bound.

The upper bound is achieved inductively, using the fact that in

$$0 \to F^rS/F^{r+1}S \to S/F^{r+1}S \to S/F^rS \to 0 \tag{4.8.2}$$

the two end-complexes are isomorphic with a shift in degree. □

The proof gives an explicit description of the ∂-closed elements in S, and thence (by change of notation) of the $\bar{\delta}$-closed elements in S_j.

4.8.3. Corollary. *The ∂-closed subalgebra of S has dimension $2^{t-1}(2^t+1)$, and is generated by the set $\{\eta_0, \ldots, \eta_{t-1}, x^r, 0 \leq r \leq 2^t - 1\}$.* □

We can now prove the conjecture of Kač [4], mentioned in the Introduction.

4.9. Proposition. $E_3(\xi_n : \mathbb{Z}) = E_\infty(\xi_n : \mathbb{Z})$. □

Proof. $d_r(\mathscr{E}_n^*) = 0$ for $r \geq 3$ because the generators of $\mathscr{E}_n^*$ have fibre-degree 1. Since $\mathscr{E}_n^*$ is a retract of $E_3(\xi_n : \mathbb{Z})$ it follows that it is a retract of every $E_r(\xi_n : \mathbb{Z})$. Also, a differential d_r can be nonzero only on the torsion subgroup $T(r)$ of $E_r(\xi_n : \mathbb{Z})$. Thus there is a spectral sequence $(T(r), d_r)$ for $r \geq 3$, of the torsion subgroups. Since $T(3) = T_n$ has exponent 2, the same is true of every $T(r)$, and, therefore, 1.4 and the inequality

$$\dim_{\mathbb{F}_2} T(\infty) \geq \dim_{\mathbb{F}_2} T(3) = \dim_{\mathbb{F}_2} T_n$$

will prove $d_r = 0$ for $r \geq 3$.

The right-hand side of this inequality was computed in Proposition 4.7. The left-hand side is $r_2(E_\infty(\xi_n : \mathbb{Z}))$ which by general nonsense is $\geq r_2(H^*(\mathrm{Spin}(2n+1) : \mathbb{Z}))$. This last integer was determined in Borel [3, formula (14.11)] and agrees with our value for the right-hand side. The case of $\mathrm{Spin}(2n)$ is similar: for SO, see 7.3. □

We now exhibit generators for $E_\infty(\xi_n : \mathbb{Z})$. First of all, the subalgebra $\mathscr{E}_n^* \subseteq E_3(\mathbb{Z}) = E_\infty(\mathbb{Z})$, and the Chow ring $A_n^*(\mathbb{Z})$ which is isomorphic to $E_3^{*,0}(\xi_n : \mathbb{Z})$ give under the product map

$$\mu : \mathscr{E}_n^* \otimes A_n^* \to E_\infty(\mathbb{Z}) \otimes E_\infty(\mathbb{Z}) \to E_\infty(\xi_n : \mathbb{Z}),$$
$$\mu(\mathscr{E}_n^* \otimes 1) = \text{free part of } E_\infty(\xi_n : \mathbb{Z}), \qquad \mu(\mathscr{E}_n^* \otimes A_n^+(\mathbb{Z})) \subseteq T_n. \tag{4.9.1}$$

One can check directly that for $n \le 4$ the second inclusion is an equality but for $n \ge 5$, it is proper[2] despite a count on the 2-ranks because there are many relations.

To get at what is missing in T_n for $n \ge 5$ we define the 'Bockstein' module $B_n \subseteq T_n$ as follows. As always, let $m = [(n+1)/2] \ge 3$.

Definition 4.10. For every subset $J \subseteq \{2, 3, \ldots, m\}$ consisting of at least *two* elements, define

$$\beta_J = \delta\left(\prod_J \rho_{2j-1}\right)$$

and $B_n^* = \mathbb{Z} - \text{span of } 1, \beta_J$ (over all admissible J).

The map $\delta : \sum_{p \ge 2} \Lambda^p(\rho_{\text{odd}}) \to B_n^+$ is an isomorphism of abelian groups, as one easily checks by reduction mod 2.

Proposition 4.11. $E_\infty(\xi_n : \mathbb{Z}) = \mu(\mathscr{E}_n^* \otimes A_n^*(\mathbb{Z})) \cdot B_n^*$.

Proof. Since $\mu : \mathscr{E}_n^* \otimes 1 \to E_\infty(\xi_n : \mathbb{Z})$ is an isomorphism onto the free part, it is equivalent to prove that T_n^* is generated by $\mu(\mathscr{E}_n^* \otimes A_n^*)$-linear combinations of β_J. Now $\delta : E_\infty(\xi_n : \mathbb{F}_2) \to T_n^*$ is surjective and using (4.6.5) we can write

$$E_\infty(\xi_n : \mathbb{F}_2) = \Lambda^*(\psi_s) \otimes A_n^*(\mathbb{F}_2) \otimes \Lambda^*(\rho_2, \ldots, \rho_n).$$

Since δ is zero on the first two factors, it suffices to show that for any subset $I \subseteq \{2, \ldots, n\}$ $\delta(\rho_I)$ lies in $\mu(\mathscr{E}_n^* \otimes A_n^*(\mathbb{Z})) \cdot B_n^*$. (Here $\rho_I = \prod_I \rho_i$.) In fact, using (4.6.2) and (4.6.3) we see that the product ρ_I in $E_\infty(\xi_n : \mathbb{F}_2)$ is itself a linear combination of products $\rho_{2j_1-1} \cdots \rho_{2j_q-1}$ with coefficients in $\mu(\mathscr{E}_n^* \otimes A_n^*(\mathbb{F}_2))$. □

There are of course many relations among the generators $\{\psi_i, \gamma_j, \beta_K\}$; and we enumerate them now as they will be required in Section 5 to prove the completeness of the corresponding set of relations for $H^*(\mathrm{Spin}(2n+1) : \mathbb{Z})$. Although the subspaces of $E_\infty(\xi_n)$ which occur below are all bigraded, this is not always essential and so we shall ignore the bigrading when convenient.

[2] See the remark after Proposition 4.13.

First, consider relations among $\{\psi_j, \gamma_k\}$; that is, the kernel of the map μ. Since $\mu \otimes \mathbb{Z}(\frac{1}{2})$ is an isomorphism, it is essentially equivalent to describe the kernel of the mod 2 reduction

$$\bar{\mu} : \bar{\mathscr{E}}_n^* \otimes A_n^*(\mathbb{F}_2) \to E_\infty(\xi_n : \mathbb{F}_2).$$

Recall from Proposition 4.8 that we can decompose $E_\infty(\xi_n : \mathbb{F}_2) = \otimes_1^m S_j \otimes \Lambda^*(\bar{\psi}_s)$. Similarly we can decompose the domain as $\otimes_1^m \Phi_j$, where $\Phi_1 = \Lambda^*(\psi_1, \dots, \bar{\psi}_s)$, and for $2 \le j \le m$

$$\Phi_j = \Lambda^*(\bar{\psi}_{2j-1}, \bar{\psi}_{2(2j-1)}, \dots, \bar{\psi}_{s_j}) \otimes \mathbb{F}_2[\bar{\gamma}_{2j-1}]/(\bar{\gamma}_{2j-1}^{a_j}),$$

$$s_j = s^{\alpha_j - 1}(2j-1).$$

Then $\bar{\mu} = \otimes_1^m \bar{\mu}_j$, where $\bar{\mu}_1 : \Phi_1 \to S_1 \otimes \Lambda^*(\bar{\psi}_s)$ is an isomorphism. Thus it is sufficient to describe each Ker $\bar{\mu}_j$, $2 \le j \le m$ and for this we can revert to the paradigm case of Lemma 4.8.1 to describe the kernel of the multiplication map

$$\tilde{\mu} : \Lambda^*(\eta_0, \dots, \eta_{t-1}) \otimes \mathbb{F}_2[x]/(x^{2^t}) \to S.$$

For convenience, put $k_0 = 2^t - 1$, $k_r = k_{r-1} - 2^r$, $1 \le r \le t-1$.

Proposition 4.12. *The kernel of $\tilde{\mu}$ is the principal ideal (ω), where*

$$\omega = \sum_{r=1}^{t-1} \eta_r \otimes x^{k_r}.$$

Proof. $\tilde{\mu}(\omega) = x^{k_0}(y_1 + xy_0) + x^{k_1}(y_2 + x^2 y_1) + \cdots$ which telescopes to zero. Now from Corollary 4.8.3 the image of $\tilde{\mu}$ is precisely the subalgebra of ∂-closed elements, which has dimension $2^{t-1}(2^t + 1)$. Hence it is enough to establish $\dim_{\mathbb{F}_2}(\omega) \ge 2^{t-1}(2^t - 1)$.

When $t = 1$, this is trivial and for $t \ge 2$ one proves that ω times the monomials in $\Lambda^*(\eta_0, \dots, \eta_{t-2}) \otimes \sum_0^{2^t-2} x^r$ are linearly independent over $\mathbb{F}_2$, giving the desired lower bound. $\square$

By change of notation we obtain Ker μ. For a fixed $j \in \{2, 3, \dots, m\}$ put $k_{0j} = 2^{\alpha_j} - 1$, $k_{ij} = k_{i-1,j} - 2^i$, $1 \le i \le \alpha_j - 1$ and $l_{ij} = 2^i(j-1)$, $0 \le i \le \alpha_j - 1$.

Proposition 4.13. Ker μ *is generated by the $m - 1$ elements*

$$\omega_j = \sum_{i=0}^{\alpha_j - 1} \psi_{l_{ij}} \otimes \bar{\gamma}_{2j-1}^{k_{ij}}, \quad 2 \le j \le m. \quad \square$$

As soon as there are at least two odd integers $\le n$—i.e., $n \ge 5$—we have $\operatorname{Im} \mu \ne E_\infty(\xi_n : \mathbb{Z})$. Here is an example:

Example. For $n=5$, the group T_5^* has 2-rank $=16(2^2-1)=48$ whereas the domain has 2-rank $=32\cdot 3=96$. But consider the ideal generated by $\omega_2=\bar{\psi}_3\otimes\bar{\gamma}_3$, $\omega_3=\bar{\psi}_5\otimes\bar{\gamma}_5$; it is a free module over $\psi_1=\Lambda^*(\psi_1,\psi_2,\psi_4)$ generated by the seven elements ω_2, ω_3, $\omega_2\bar{\gamma}_5$, $\bar{\psi}_5\omega_2$, $\bar{\psi}_5\omega_2\bar{\gamma}_5$, $\omega_3\bar{\gamma}_3$ and $\bar{\psi}_3\omega_3$ (because $\bar{\psi}_5\omega_2\bar{\gamma}_5=\bar{\psi}_3\omega_3\bar{\psi}_3$). Hence the image of μ has 2-rank 40. The eight missing elements are precisely $\Lambda^*(\psi_1,\psi_2,\psi_4)\cdot\beta_{(2,3)}$. □

Put $\mathcal{S}^{*,*}=\operatorname{Im}\mu\subseteq E_\infty(\xi_n:\mathbb{Z})$. We will now analyze the $\mathcal{S}^{*,*}$-linear relations among the β_J. To this end we record a mild extension of Proposition 4.13, namely, the kernel of the multiplication map

$$\mu:\bar{\mathscr{E}}_n^*\otimes A_n^*(\mathbb{F}_2)\otimes\Lambda^*(\rho_{\text{odd}})\to E_\infty(\xi_n:\mathbb{F}_2)\,. \tag{4.13.1}$$

As before we decompose the range into $\otimes_1^m S_j\otimes\Lambda^*(\bar{\psi}_s)$ and the domain as $\otimes_1^m\mathcal{S}_j$ where

$$\mathcal{S}_1=\Phi_1\,,\qquad \mathcal{S}_j=\Phi_j\otimes\Lambda^*(\rho_{2j-1})\quad 2\le j\le m\,.$$

Then $\mu'=\otimes_1^m\mu_j'$, $\mu_1'=\mu_1$ is an isomorphism and each μ_j' is surjective (from the last paragraph in the proof of Proposition 4.11). We are again reduced to our paradigm example, to compute the kernel of the *surjection*

$$\tilde{\mu}':\Lambda^*(y_0)\otimes\Lambda^*(\eta_0,\ldots,\eta_{t-1})\otimes\mathbb{F}_2[x]/(x^{2^t})\to S\,.$$

Define

$$\zeta=\sum_{r=0}^{t-1}\eta_r\otimes x^{k_r-1}+y_0\otimes x^{k_0}$$

where the k_r's are as in Proposition 4.12. Counting dimensions and noting that $\zeta x=\omega$ it follows easily that $\operatorname{Ker}\tilde{\mu}'=(\zeta)$. Hence in the notation above we have the following proposition:

Proposition 4.14. *The kernel of μ' is generated by the $m-1$ elements*

$$\zeta_j=\sum_{i=0}^{j-1}\psi_{1_{ij}}\otimes\bar{\gamma}_{2j-1}^{k_{ij}-1}+\bar{\gamma}_{2j-1}^{aj-1}\otimes\rho_{2j-1}\quad 2\le j\le m\,.\qquad\square$$

To describe the kernel of

$$\nu:\mathcal{S}^*\otimes B_n^*\to E_\infty(\xi_n:\mathbb{Z}) \tag{4.14.1}$$

it is useful to write $\zeta'=$ image of ζ_j in $\mathcal{S}^*\otimes\Lambda^*(\rho_{\text{odd}})$ so that $\zeta_j'=s_j\otimes 1+\zeta_j''$ where $\zeta_j''=\bar{\gamma}_{2j-1}^{aj-1}\otimes\rho_{2j-1}$. Notice that $\bar{\nu}(\zeta_j'')$ in $E_\infty(\xi_n:\mathbb{F}_2)$ is the reduction of an

integral class, and the fact that $\bar{\nu}(\zeta_j)=0$ gives a formula for $\bar{\nu}(\zeta_j'')$ as an element of $\mathcal{S}^*$.

The restriction of ν to $\mathcal{S}^*\otimes 1$ is injective by definition; and, therefore, every element of $\operatorname{Ker}\nu$ is determined by its projection to $\mathcal{S}^*\otimes B_n^+$. Since $\delta:\sum_{p\geq 2}\Lambda^p(\rho_{\mathrm{odd}})\to B_n^+$ is an isomorphism (see Definition 4.10) consider the following diagram:

$$\begin{tikzcd}
 & \mathcal{S}^*\otimes\sum\limits_{p\geq 2}\Lambda^p(\rho_{\mathrm{odd}}) \arrow[rr, "1\otimes\delta", "\sim"'] \arrow[d, "\mu''"] & & \mathcal{S}^*\otimes B_n^+ \arrow[d, "\nu"] & \\
0 \arrow[r] & E_\infty(\xi_n:\mathbb{Z})\otimes\mathbb{F}_2 \arrow[r] & E_\infty(\xi_n:\mathbb{F}_2) \arrow[r] & \operatorname{Tor}(E_\infty(\xi_n:\mathbb{Z}),\mathbb{F}_2) \arrow[r] & 0
\end{tikzcd} \tag{4.14.2}$$

where μ'' = restriction of μ' and in the last vertical we have identified $\operatorname{Tor}(A,\mathbb{F}_2)$ with the subgroup of element of order 2 in A. Thus we have to compute $\mu''^{-1}(E_\infty(\xi_n:\mathbb{Z})\otimes\mathbb{F}_2)$.

Proposition 4.15. *As a module over $\bar{\mathcal{S}}^*$ the image of μ'' is generated by β_I, $|I|\geq 3$.*

Proof. Clearly $\operatorname{Im}\mu''$ lies in the intersection of $\sum_{p\geq 2}(E^{*,p}(\xi_n:\mathbb{Z})\otimes\mathbb{F}_2)$ and the ideal $E_\infty(\xi_n:\mathbb{F}_2)$ generated by $\Lambda^2(\rho_{\mathrm{odd}})$. Now as a module over $\bar{\mathcal{S}}^*$ the space $\sum_{p\geq 2}E^{*,p}(\xi_n:\mathbb{Z})\otimes\mathbb{F}_2$ is generated by elements of two types: β_I with $|I|\geq 3$ or $\bar{\psi}_j\beta_K$ with $|K|\geq 2$ (see Proposition 4.11). The second type will lie in the ideal generated by $\Lambda^2(\rho_{\mathrm{odd}})$ iff j is odd *and* $>n/2$; and in this case $\bar{\psi}_j\beta_K=0$ if $j\in K$. If $j=2i-1>n/2$ is not in K, the fact that $\bar{\gamma}_{2i-1}^2=0$ implies $\bar{\psi}_j\beta_K=\gamma_{2i-1}\beta_{K\cup\{i\}}$. □

For the next proposition *only* introduce the abbreviation $I'=\{2i-1\mid i\in I\}$ for $I\subseteq\{2,3,\ldots,m\}$. (Caution: I' will have a different meaning in Section 6.) Thus $\rho_{I'}=\prod_I\rho_{2i-1}$.

Proposition 4.16. *As a module over $\bar{\mathcal{S}}^*$, $\mu''^{-1}(E_\infty(\xi_n:\mathbb{Z})\otimes\mathbb{F}_2)$ is generated by*
(a) *if $|I|\geq 3$, $I_i=I-\{i\}$, $\sum_\infty\gamma_{2i-1}\otimes\beta_{I_i'}$,*
(b) *if $|K|\geq 2$, $\zeta_j\otimes\rho_K$.*

Proof. In view of Proposition 4.15 it is sufficient to observe that the $\bar{\mathcal{S}}^*$-module $\operatorname{Ker}\mu''$ is generated by elements of type (b): because $\mu''(\text{type(a)})=\beta_I$. □

Proposition 4.17. *As a module over $\mathcal{S}^*$,* Ker ν *is generated by*
(a) *if $|I|\geq 3$, and $I_i=I-\{i\}$ $\sum_I\gamma_{2i-1}\otimes\beta_{I_i}$,*
(b) *if $j\in I$, $s_j\otimes\beta_I$ (see Proposition* 4.14),
(c) *if $j\not\in I$, $s_j\otimes\beta_I+\gamma_{2j-1}^{aj-1}\otimes\beta_{I\cup\{j\}}$, where if I is a singleton, then $\beta_I=\gamma_{2i-1}$.*

Proof. These generators correspond under $(1\otimes\delta)$ in (4.14.2) to types (a) and (b)

of Proposition 4.16; in fact, (a) is just a restatement of $\bar{\delta}(\beta_I) = 0$. Types (b) and (c) arise according as $j \in K$ or $j \notin K$ in Proposition 4.16: if $j \in K$, then $\zeta_j'' \otimes \rho_K = 0$ which reduces $\zeta_j \otimes \rho_K$ to type (b) above, whereas if $j \notin K$, then $\zeta_j'' \otimes \rho_K = \gamma_{2j-1}^{aj-1} \otimes \rho_{K \cup \{j\}}$. □

For completeness we give the product relations among the β_J, although they are not needed for the proof of Proposition 5.8.

Proposition 4.18. (a) *If $I \cap J$ has at least two elements, then $\beta_I \beta_J = 0$.*

(b) *If $I \cap J = \{k\}$, $\beta_I \beta_J = \gamma_k \beta_{I \cup J}$.*

(c) *If $I \cap J = \phi$, let $I = \{i_1, \ldots, i_p\}$ and put $K_r = I \cup J - \{i_r\}$, $1 \leq r \leq p$. Then $\beta_I \beta_J = \sum_1^p \gamma_{i_r} \beta_{K_r}$.* □

It is not difficult to convince oneself that the set $\{1, \psi_i, \gamma_j, \beta_K\}$ forms a minimal set of generators for the ring $E_\infty(\xi_n : \mathbb{Z})$; we omit the proof.

5. The integral cohomology ring of Spin(2n + 1)

Recall that the functor

$$Q(X) = H^*(X : Z)/\mathrm{Tors}\, H^*(X : Z)$$

evaluated on Spin$(2n+1)$ is an exterior algebra on n generators of degree $4j - 1$, $1 \leq j \leq n$ (see Section 2). For this section we introduce the abbreviations

$$H^*(n : R) = H^*(\mathrm{Spin}(2n+1) : R)\,, \qquad \mathcal{T}_n^* = \mathrm{Tors}\, H^*(n : \mathbb{Z})$$

so that we have an exact sequence

$$0 \to \mathcal{T}_n^* \to H^*(n : \mathbb{Z}) \xrightarrow{q} \Lambda^*(y_1, \ldots, y_n) \to 0 \tag{5.1}$$

and we will produce generators for the ring $H^*(n : \mathbb{Z})$ which are compatible with (5.1). Since $\mathcal{T}_n^*$ is (additively) of exponent 2, it injects into $H^*(n : \mathbb{F}_2)$ and we can work out the relations among its generators by reducing mod 2, once we have a workable description of $H^*(n : \mathbb{F}_2)$. We have already seen that $\mathcal{T}_n^* = 0$ for $n = 1, 2$ (Proposition 4.7); henceforth we shall assume $n \geq 3$.

Lemma 5.2. *$E^{2k,1}(\xi_n : R)$ injects into $H^{2k+1}(n : R)$ and the multiplication*

$$E^{2k,1}(\xi_n : R) \otimes E^{2l,0}(\xi_n : R) \to E^{2(k+l),1}(\xi_n : R)$$

is compatible with cup-product in $H^*(n : R)$.

Proof. $E_2(\xi_n : R)$ is zero when the base degree is odd, and hence the same is true of $E_\infty(\xi n : R)$. Thus if F is the filtration on $H^*(n : R)$ induced from ξ_n, then for all p, q $(q \geq 2p+1)$

$$F^{2p+1}(H^q(n : R)) = F^{2p+2}H^q(n : R).$$

Putting $p = k$, $q = 2k+1$ we see

$$F^{2k+1}H^{2k+1}(n : R) = 0$$

and so $E^{2k,1}(\xi_n : R) = F^{2k}H^{2k+1}(n : R) \subseteq H^{2k+1}(n : R)$. The second assertion follows from the compatibility of F^j with products. □

Hence we can use 4.11 to define some generators of $H^*(n : \mathbb{Z})$. Put

$$x_j \in H^{4j-1}(n : \mathbb{Z}) = \text{the image of } \psi_j \in E^{4j-2,1}(\xi_n : \mathbb{Z}) \tag{5.2.1}$$

and

$$\tilde{\mathscr{E}}_n^* = \text{subalgebra generated by } \{1, x_1, \ldots, x_n\} \text{ in } H^*(n : \mathbb{Z}).$$

(For Spin$(2n+2)$, we also include the image of $\tau_{n+1} \in E^{2n,1}(\mathbb{Z})$, which is $\pm\varepsilon$ (see 2.2.1) as one sees from (4.2.1); and then the analogues of Proposition 5.3 and (5.2.2) go through.) Then $x_ix_j = -x_jx_i$ for reasons of degree, so $x_i^2 \in \mathscr{T}_n^*$. On the other hand, the element $x_1 \cdots x_n$ maps to $\psi_1 \cdots \psi_n$ in $E_\infty(\xi_n : \mathbb{Z})$ which is a generator of the E_∞-group in total deg $=$ dim Spin$(2n+1)$. Hence $x_1 \cdots x_n$ is a generator of $H^{\text{top}}(n : \mathbb{Z})$ and so in (5.1)

$$q(x_1 \cdots x_n) = y_1 \cdots y_n \tag{5.2.2}$$

whence we have the following:

Proposition 5.3. *$\tilde{\mathscr{E}}_n^* \xrightarrow{i} H^*(n : \mathbb{Z}) \xrightarrow{\pi} \Lambda^*(y_1, \ldots, y_n)$ is surjective.*

Proof. Rationally (or even over $\mathbb{Z}(\frac{1}{2})$) the maps i, q are isomorphisms, so $q \circ i$ has finite cokernel. Equation (5.2.2) coupled with the appropriate analogue of 1.6 shows that the image of $q \circ i$ is a retract of $\Lambda^*(y_1, \ldots, y_n)$. □

Corollary 5.3.1. *$\tilde{\mathscr{E}}_n^*$ is the free part of $H^*(n : \mathbb{Z})$, and has as a $\mathbb{Z}$-basis the set*

$$\{1, x_{i_1} \cdots x_{i_p} \mid 1 \leq i < \cdots < i_p \leq n\}. \qquad \square$$

Coming now to the torsion ideal, set $u_i \in H^{2i}(n : \mathbb{Z})$ to be the image of

$\gamma_i \in E_\infty^{2i,0}(\xi_n : \mathbb{Z}) = A_n^{2i}(\mathbb{Z})$. Since $E_\infty^{*,0}(\xi_n : \mathbb{Z})$ is a subring of the cohomology we know

$$u_i^2 = u_{2i} \quad \text{if } 1 \le i \le n/2\,, \qquad u_i^2 = 0 \quad \text{for } n/2 < i \le n\,.$$

Finally, we will define elements $v_I \in \mathcal{T}_n^*$ corresponding to the Bockstein generators β_I of Definition 4.10, which are specified by their mod 2 reductions. For this purpose we need to give a suitable set of algebra generators of $H^*(n : \mathbb{F}_2)$; and at the same time we will complete the description of this latter ring as a module over $\mathcal{A}(2)$.

Using Lemma 5.2 and the description of $E_\infty(\xi_n : \mathbb{F}_2)$ in Proposition 4.6, define

$$\begin{array}{ll} z_{2i-1} \in H^{2i-1}(n : \mathbb{F}_2) & \text{as the image of } \rho_i \in E_\infty^{2i-2,1}(\xi_n : \mathbb{F}_2)\,,\ 2 \le i \le n\,, \\ z_{2j} \in H^{2j}(n : \mathbb{F}_2) & \text{as the image of } \bar{\gamma}_j \in E_\infty^{2j,0}(\xi_n : \mathbb{F}_2)\,, \\ & \quad \text{where } 3 \le j \le n\,,\ j \ne 2^k\,, \\ z \in H^{2a_1-1}(n : \mathbb{F}_2) & \text{as the image of } t \otimes \bar{\gamma}_1^{a_1-1} \in E_\infty^{2a_1-2,1}(\xi_n : \mathbb{F}_2)\,. \end{array}$$

Proposition 5.4. (a) $H^*(n : \mathbb{F}_2) = \Delta_{\mathbb{F}_2}(z_j, z), 3 \le j \le 2n,\ j \ne 2^i$.

(b) $\mathrm{Sq}^i(z_j) = \binom{j}{i} z_{i+j}$ *if* $3 \le i + j \le 2n$, *and* $i + j \ne 2^k$, $i \le j$; $\mathrm{Sq}^i(z_j) = 0$ *otherwise.*

(c) $\mathrm{Sq}^i(z) = 0$ *for* $i \ge 2$ *and* $\mathrm{Sq}^1(z) = 0$ *if* $n = 2^\alpha$. $\mathrm{Sq}^1(z) = \sum_{2s-n}^{s-1} z_{2j} z_{4s-2j}$ *if* $n \ne 2^\alpha$.

Proof. Part (a) follows from the fact that $\{\rho_i, \gamma_j, t \otimes \bar{\gamma}_1^{a_1-1}\}$ form a simple system of generators for $E_\infty(\xi_n : \mathbb{F}_2)$. The equations in part (b) are the same as those of Borel [3, Theorem 12.1(c)] and they will be proved by showing that the classes z_i are the pullbacks from $H^*(\mathrm{SO}(2n+1) : \mathbb{F}_2)$ of elements which transgress to the universal Stiefel–Whitney classes. Part (c) will follow from (4.6.5) once we identify z with $\pi^*(x)$ where x is the element defined in 3.9.

As these assertions follow from (sometimes elaborate) diagram chases, their proofs are deferred to the end of the section. □

For some purposes it is useful to have another generator replacing z. In (5.4.7) we have shown

$$\bar{x}_s = z \quad \text{if } n = s\,, \qquad \bar{x}_s = z \bmod (z_3, \ldots, z_{2n}) \quad \text{if } n = s + r\,,\ 1 \le r < s\,.$$

Corollary 5.4.4. $H^*(n : \mathbb{F}_2) = \Delta_{\mathbb{F}_2}(z_j, \bar{x}_s),\ 3 \le j \le 2n,\ j \ne 2^i$. □

The point here is that $\mathrm{Sq}^1(\bar{x}_s) = 0$ and $\bar{x}_s^2 = 0$; although of course this now complicates $\mathrm{Sq}^i(\bar{x}_s)$ for $2 \le i \le 4s - 1$.

The analogue of Proposition 5.4 holds for $H^*(\mathrm{Spin}(2n+2) : \mathbb{F}_2)$ with the z_j running from z_3 to $z_{2n+1} = \bar{\tau}_{n+1}$, $j \ne 2^i$. Moreover, using the isomorphism $X_n' \simeq$

Y'_{n+1} (see 3.9.2) the exceptional generator z in $H^*(\mathrm{Spin}(2n+2):\mathbb{F}_2)$ restricts to the corresponding element in $H^*(n:\mathbb{F}_2)$ and in particular we have $\mathrm{Sq}^1(z)=0$ in $H^*(\mathrm{Spin}(2n+2):\mathbb{F}_2)$ iff $n=2^\alpha$.

Corollary 5.4.5 The 'obvious' map $H^*(\mathrm{Spin}(2n+1):\mathbb{F}_2)\to H^*(\mathrm{Spin}(2n+2):\mathbb{F}_2)$ given by $z_i\mapsto z_i$, $z\mapsto z$ is a multiplicative splitting for the restriction map and so

$$H^*(\mathrm{Spin}(2n+2):\mathbb{F}_2)\simeq H^*(\mathrm{Spin}(2n+1):\mathbb{F}_2)\otimes\Lambda^*(z_{2n+1})\,. \qquad \square$$

From the remark preceding Definition 4.10 we see that for $n\le 4$ the ring $H^*(n:\mathbb{Z})$ is generated by $x_1,\dots,x_n$ and u_j. The new generators v_I arise only for $n\ge 5$ and are defined as follows. Put $m=[(n+1)/2]$ and $I\subseteq\{2,3,\dots,m\}$ a subset of at least *two* elements. The element v_I is specified by

$$\bar v_I=\mathrm{Sq}^1\left(\prod_I z_{4i-3}\right) \tag{5.4.6}$$

One checks easily that the v_I map to the elements $\beta_I\in E_\infty(\xi_n:\mathbb{Z})$ of Definition 4.10 and hence by Proposition 4.11, $H^*(n:\mathbb{Z})=$ subring generated by $(\bar{\mathscr{E}}_n^*\otimes A_n^*(\mathbb{Z}))\cdot V_n$, where V_n is the $\mathbb{Z}$-span of $\{1,v_I\}$.

We first compute the subring generated by $\bar{\mathscr{E}}_n^*$, $A_n^*(\mathbb{Z})$. Since x_j^2 is a 2-torsion element, it can be calculated from $\bar x_j^2$. Now from Lemma 5.2 and (4.6.3) to (4.6.5),

If $1\le j\le n/2$,

$$\bar x_j=\begin{cases} z_{4j-1}+z_{2j-1}z_{2j}\,, & \text{if } j\ne 2^i\,,\\ z_{4j-1}\,, & \text{if } j=2^i\,.\end{cases} \tag{5.4.7}$$

If $n/2<j\le n$,

$$\bar x_j=z_{2j-1}z_{2j}\,,\quad \text{if } j\ne s\,,$$

$$\bar x_s=z+\sum_{j=2s-n}^{s-1} z_{2j-1}z_{4s-2j}$$

and of course $\bar u_i=z_{2i}$.

Proposition 5.5. (a) *If* $1\le j\le (n+1)/4$,

$$x_j^2=\begin{cases} u_{4j-1}+u_{2j-1}u_{2j}\,, & \textit{for } j\ne 2^i\,,\\ u_{4j-1}\,, & \textit{for } j=2^i\,.\end{cases}$$

(b) *If* $(n+1)/4\le j\le n/2$,

$$x_j^2 = \begin{cases} u_{4j-1}\,, & \text{for } j = 2^i\,, \\ u_{2j-1}u_{2j}\,, & \text{for } j \neq 2^i\,, \\ 0\,, & \text{for } j = 2^i\,. \end{cases}$$

(c) *If* $n/2 < j \leq n$,

$$x_j^2 = 0\,.$$

Proof. We can reduce mod 2 to check these equations, which then follow from (5.4.7), except possibly for x_s when $s < n$; but $z^2 = 0$ by Proposition 5.4 and each summand of $(\bar{x}_s - z)$ has a factor z_k with $k > n$, whence $x_s^2 = 0$. □

There is a further relation among the products $x_j u_k$, which comes from Proposition 4.13. Using the notation $k_{ij} = 2^{\alpha_j} - \sum_0^i 2^r$ and $l_{ij} = 2^i(2j-1)$ for $2 \leq j \leq m$ and $0 \leq i \leq \alpha_j - 1$, we have

$$\sum_{i=0}^{\alpha_j - 1} x_{l_{ij}} u_{2j-1}^{k_{ij}} = 0\,. \tag{5.5.1}$$

Proposition 5.6. *The subring* S^* *of* $H^*(n : \mathbb{Z})$ *generated by* $\{1, x_j, u_k\}$ *is isomorphic to* $\mathbb{Z}[x_1, \ldots, x_n] \otimes A_n^*(\mathbb{Z})$ *subject only to Proposition* 5.5 *and* (5.5.1).

Proof. We only need to show that these relations (and the relations defining $A_n^*(\mathbb{Z})$, of course) generate the full ideal of relations. So consider a polynomial $P(u_i, x_j)$ which is zero. Using Proposition 5.5 we can reduce P to an $A_n^*(\mathbb{Z})$-linear combination of square-free monomials in the x_i and then from Corollary 5.3.1 we see that the coefficients must in fact lie in $A_n^+(\mathbb{Z})$. Now P will give rise in $\operatorname{gr} H^*(n : \mathbb{Z}) = E_\infty(\xi_n : \mathbb{Z})$ to $A_n^*(\mathbb{Z})$-linear relations among the products $\psi_I \gamma_J$, but these have already been listed in Proposition 4.13, and they correspond precisely to (5.5.1). □

We mentioned at the end of Section 4 that $\{\psi_k \gamma_{2j-1}, \beta_I\}$ form a minimal set of generators for $E_\infty(\xi_n : \mathbb{Z})$. From that statement and Proposition 5.5 we deduce that $\{1, x_k, u_{4j-3}, v_I\}$ form a minimal set of generators for $H^*(n : \mathbb{Z})$.

We now compute the S^*-linear relations in V_n. In analogy with the discussion preceding Proposition 4.15 define

$$\tilde{s}_j = \sum_{i=0}^{\alpha_j - 1} x_{l_{ij}} u_{2j-1}^{k_{ij}-1}\,,$$

where $2 \leq j \leq m$, $l_{ij} = 2^i(2j-1)$ and $k_{ij} = 2^{\alpha_j} = \sum_0^i 2^r$. Then one has the following:

Proposition 5.7. *As an* S^**-module, the kernel of the multiplication map*

$S^* \otimes V_n \to H^*(n:\mathbb{Z})$ *is generated by*

(a) *If* $|I| \geq 3$, $I_i = I - \{i\}$, $\sum_I u_{2i-1} \otimes v_{I_i}$,

(b) *If* $j \in I$, $\tilde{s}_j \otimes v_I$,

(c) *If* $j \not\in I$, *then* $\tilde{s}_j \otimes v_I + u_{2j-1}^{a_j-1} \otimes v_{i\cup\{j\}}$, *where for a singleton* $I = \{i\}$, $v_I = u_{2i-1}$.

Proof. We reduce mod 2 to verify that these elements are in the kernel. For the completeness, we can throw the brunt of the work back on Proposition 4.17 as was done in the proof of Proposition 5.6. From a computational, 'hands-on' viewpoint the only difference between the generators x_i, u_j, v_K and ψ_i, γ_i, β_K is that in the latter set $\psi_j^2 = 0$, $\beta_K^2 = 0$, which is not necessarily so in the former set. However, the generators of the former set still satisfy the analogous linear independence statements, and this is really all that was used in the proof of Proposition 4.17. □

All we have to do now to complete the description of the ring $H^*(n:\mathbb{Z})$ is to describe the products $v_I v_J$ as S^*-linear combinations of v_K. Here we get formulae which are slightly different from those of Proposition 4.18. First of all, $v_I^2 \in A_n^+(\mathbb{Z})$ is not zero in general: but its value is deducible from Proposition 5.8(a) or alternatively easily described using $z_{2j-1}^2 = \bar{u}_{2j-1}$. For index sets I, J we give below two product formulae which give the general formula by an obvious induction.

Proposition 5.8. (a) *If* $K = I \cap J$, *put* $I' = I - K$, $J' = J - K$, *then*

$$v_I v_J = \Big(\prod_K u_{4k-3}\Big) v_{I'} v_{J'} + \Big(\prod_K u_{2k-1}\Big) v_{I'\cup J}\,,$$

(b) *If* $I \cap J = \emptyset$, *write* $I = \{i_1, \ldots, i_p\}$ *and* $K_r = I \cup J - \{i_r\}$, $1 \leq r \leq p$; *then*

$$v_I v_J = \sum_1^p u_{2i_r-1} v_{K_r}\,.$$

Proof. We can work in $H^*(n:\mathbb{F}_2)$, and then the verifications are routine. For part (a) $z_I = \prod_I z_{4i-3}$, $z_I = z_{I'} \prod_K z_{4k-3}$ and $z_J = z_{J'} \prod_K z_{4k-3}$ and then compute $\mathrm{Sq}^1(z_{I'} \prod z_{4k-3})$, $\mathrm{Sq}^1(z_{J'} \prod z_{4k-3})$. For part (b) use induction on $p = |I|$. □

Theorem 2. $H^*(\mathrm{Spin}(2n+1):\mathbb{Z})$ *is generated by* $\{1, x_i, u_j, v_K\}$ *subject to the relations in Propositions* 5.5 *to* 5.8. *The free part consists of all square-free monomials in* x_i; *all squares lie in the Chow ring* $A_n^*(\mathbb{Z})$ *which is generated by* $\{1, u_j\}$. □

Here is an illustration of the theorem for Spin(11).

Example. $n = 5$ for Spin(11) and so the free part of $H^*(\mathrm{Spin}(11) : \mathbb{Z})$ is generated by the square-free monomials in $x_1, \ldots, x_5$; $\deg x_j = 4j - 1$. From Proposition 5.5 we have $x_1^2 = u_3$, and $x_i^2 = 0$ for $i \geq 2$. The Chow ring is $\Lambda^*(u_3, u_5)$ with $2u_i = 0$. The only Bockstein element is v in degree 15, $\bar{v} = z_6 z_9 + z_5 z_{10}$.

The relations from Proposition 5.6 are $x_3 u_3 = 0$, $x_5 u_5 = 0$. The relations from Proposition 5.7 are only of types (b), (c) because $m = 3$; and they are $x_3 v = 0$ for type (b) and $u_5 v + u_3 x_3$, $u_3 v + u_5 x_3$ for type (c). From Proposition 5.8, $v^2 = 0$. Thus, using $x_1^2 = u_3$ and $u_3^2 = 0$

$$H^*(\mathrm{Spin}(11) : \mathbb{Z}) = \mathbb{Z}[x_1]/(x_1^4) \otimes \Lambda_{\mathbb{Z}}^*(x_2, \ldots, x_5; u_5, v)$$

modulo the ideal

$$(2u_5, 2v, x_3 x_1^2, x_5 u_5, x_3, v, x_5 v, u_5 v + x_1^2 x_5, x_1^2 v + u_5 x_3)\,.$$

We now complete the proof of Proposition 5.4 in the following three lemmas, the first of which establishes the analogue of Proposition 5.4(a) and (b) for the group $\mathrm{SO}(2n+1)$. From 7.2 we see that $E_\infty(\xi_n' : \mathbb{F}_2)$ satisfies the analogue of Lemma 5.2 and hence there are elements

$$h_{2i-1} = \text{Image of } \tau_i \in E_\infty^{2i-2,1}(\xi_n' : \mathbb{F}_2) \to H^{2i-1}(\mathrm{SO}(2n+1) : \mathbb{F}_2)\,,$$
$$h_{2i} = \text{Image of } \bar{\gamma}_i \in E_\infty^{2i,0} \subseteq H^{2i}(\mathrm{SO}(2n+1) : \mathbb{F}_2)\,.$$

Lemma 5.4.1. (a) $H^*(\mathrm{SO}(2n+1) : \mathbb{F}_2) = \Delta_{\mathbb{F}_2}^*(h_1, \ldots, h_{2n})$,

(b) $\mathrm{Sq}^i(h_j) = \binom{j}{i} h_{i+j}$ *if* $1 \leq i + j \leq 2n$, $\mathrm{Sq}^i(h_j) = 0$, *otherwise.*

Proof. Part (a) is clear since $\{\tau_i, \gamma_j\}$ forms a simple set of generators for $E_\infty(\xi_n' : \mathbb{F}_2)$. From the Borel transgression theorem and the Wu formulae the elements $g_i \in H^i(\mathrm{SO}(2n+1) : \mathbb{F}_2)$ which transgress to the universal Stiefel–Whitney classes satisfy (b). We will show that $g_i = h_i$, $1 \leq i \leq 2n$.

First of all, counting dimensions over $\mathbb{F}_2$ using part (a), 1.4 implies that the mod 2 spectral sequence of

$$\mathrm{SO}(2n-1) \xrightarrow{i} \mathrm{SO}(2n+1) \xrightarrow{\pi} V(n) = V(n, 2)$$

collapses at E_2. Since $H^*(V(n) : \mathbb{F}_2) = \Lambda^*(v_{2n-1}, v_{2n})$ it follows that $h_j = g_j$ if $i^*(h_j) = i^*(g_j)$, $1 \leq j \leq 2n-2$, and $h_k = \pi^*(v_k)$ if $k = 2n-1, 2n$. Proceeding inductively, it is only necessary to verify the assertion for $n = 1$ and to prove $h_k = \pi^*(v_k)$, $k = 2n-1, 2n$.

Now for $n = 1$ the nonzero groups in $H^*(\mathrm{SO}(3) : \mathbb{F}_2)$ are all 1-dimensional, so $h_i = g_i$, $1 \leq i \leq 2$. Assuming the result for $n-1$, consider the diagram

$$\begin{array}{ccccc} U(n-1) & \rightarrow & SO(2n-1) & \rightarrow & X_{n-1} \\ \downarrow & & \downarrow & & \downarrow \\ U(n) & \longrightarrow & SO(2n+1) & \longrightarrow & X_n \\ \downarrow & & \downarrow & & \downarrow \\ S^{2n-1} & \longrightarrow & V(n) & \longrightarrow & S^{2n} \end{array}$$

Now v_{2k-1} restricts to a generator of $H^*(S^{2n-1} : \mathbb{F}_2)$ which pulls back to τ_n (see comment following Proposition 4.2) and v_{2n} comes from $[S^{2n}]$ which pulls up to γ_n by definition. The rest follows. $\square$

The first Stiefel–Whitney class of the $\mathbb{Z}/2$-bundle $p : \mathrm{Spin}(2n+1) \rightarrow SO(2n+1)$ is h_1, so from the mod 2 Gysin sequence

$$\mathrm{Im}(p^*) \simeq H^*(SO(2n+1) : \mathbb{F}_2)/(h_1)\,.$$

Since $h_j^2 = h_{2j}$, for $j \leq n$ (see Lemma 5.4.1(b)) we have the following:

Lemma 5.4.2. *$p^*(h_j) = 0$ if $j = 2^i$; $p^*(h_j) = z_j$, $3 \leq j \leq 2n$, $j \neq 2^i$.*

Proof. There is a map of bundles $p : \xi_n \rightarrow \xi_n'$ which is the identity on the base, and the given double covering on the fibre and total space. From the explicit constructions of Proposition 4.6 and 7.2,

$$p^* : E_\infty(\xi_n' : \mathbb{F}_2) \rightarrow E_\infty(\xi_n : \mathbb{F}_2)$$

is given by

$$p^*(\tau_1) = p^*(\bar{\gamma}_1) = 0\,, \qquad p^*(\tau_j) = \rho_j\,, \quad p^*(\bar{\gamma}_j) = \bar{\gamma}_j\,, \qquad 2 \leq j \leq n\,,$$

from which the result follows by 5.2. $\square$

Recall the map $\mathrm{Spin}(2n+1) \xrightarrow{\pi} X_n'$ of Section 3 and the element $x \in H^{2a_1-1}(X_n' : \mathbb{F}_2)$ of Proposition 3.7.

Lemma 5.4.3. *$\pi^*(x) = z$ and $\mathrm{Sq}^1(z) = \sum z_{2j} z_{4s-2j}$, $2s - n \neq 2^i$, $j \neq 2^k$.*

Proof. The d_2-cocycle $t \otimes \bar{\gamma}_1^{a_1-1} \in E_2(\xi_n : \mathbb{F}_2)$ can also be defined in the E_2-term of the spectral sequence for the following bundle. Let $\tilde{U}(n) = p^{-1}(U(n))$, p as in Lemma 5.4.1(a). A trivial induction on n shows that

$$H^*(U(n) : \mathbb{Z}) = \Lambda^*(b_1, \ldots, b_n)\,, \qquad H^*(\tilde{U}(n) : \mathbb{Z}) = \Lambda^*(\tilde{b}_1, \ldots, \tilde{b}_n)\,,$$

where the b_j's transgress to the universal Chern classes and

$$p^*(b_1) = 2\tilde{b}_1\,, \qquad p^*(b_j) = \tilde{b}_j\,, \quad 2 \le j \le n\,.$$

Hence in the $\mathbb{F}_2$-spectral sequence of $\tilde{U}(n) \to \mathrm{Spin}(2n+1) \to X_n$ we have $d_2(\tilde{b}_1) = \bar{\gamma}_1$, $d_2(\tilde{b}_j) = 0$ for $2 \le j \le n$ and so the E_3-term of this spectral sequence is

$$\Lambda^*_{\mathbb{F}_2}(\tilde{b}_2, \dots, \tilde{b}_n) \otimes A^*_n(\mathbb{F}_2) \otimes \Lambda^*(\tilde{b}_1 \otimes \bar{\gamma}_1^{a_1-1})\,.$$

Counting dimensions and comparing with Proposition 4.6 this is E_∞ as well. Now considering the diagram

$$\begin{tikzcd} T \arrow[r] \arrow[d, "i"'] & \mathrm{Spin}(2n+1) \arrow[r] \arrow[d, equal] & F(2n+1) \arrow[d] \\ \tilde{U}(n) \arrow[r] & \mathrm{Spin}(2n+1) \arrow[r] & X_n \end{tikzcd}$$

shows easily that $i^*(\tilde{b}_1) = t$, whence $\tilde{b}_1 \otimes \bar{\gamma}^{a_1-1}$ in the E_3-term of the second row goes over into $t \otimes \bar{\gamma}^{a_1-1}$ in $E_3(\xi_n : \mathbb{F}_2)$, and hence $\tilde{b}_1 \otimes \bar{\gamma}_1^{a_1-1}$ maps to z under $E^{2a_1-2,1} \to H^{2a_1-1}(n : \mathbb{F}_2)$.

We also know from Propositions 3.8 and 4.6 that the mod 2 spectral sequence of

$$\mathrm{SU}(n) \to \mathrm{Spin}(2n+1) \to X'_n$$

collapses at E_2. Performing a diagram chase of the bundles

$$\begin{tikzcd} \mathrm{SU}(n) \arrow[r] \arrow[d] & \mathrm{Spin}(2n+1) \arrow[r] \arrow[d] & X'_n \arrow[d] \\ \tilde{U}(n) \arrow[r] & \mathrm{Spin}(2n+1) \arrow[r] & X_n \end{tikzcd}$$

we obtain $\pi^*(x) = z$. For $\mathrm{Sq}^1(z)$ we note that $\mathrm{Sq}^1(\bar{x}_s) = 0$ because $\bar{x}_s$ is the reduction of an integral class. Hence formula (5.4.7) gives the result. $\square$

From 3.9.1 we see that when there is no power of 2 intervening between n_1 and n_2, $z \in H^{2a_1-1}(n_2 : \mathbb{F}_2)$ restricts to $z \in H^{2a_1-1}(n_1 : \mathbb{F}_2)$; and it is this feature which makes it amenable for calculations.

6. The integral Pontryagin ring of Spin(k)

We begin by calculating the comultiplication on our generators in $H^*(\mathrm{Spin}(2n+1) : \mathbb{F}_2)$ induced by the product map

$$\mu : \mathrm{Spin}(2n+1) \times \mathrm{Spin}(2n+1) \to \mathrm{Spin}(2n+1)\,.$$

The case of even k is then an immediate consequence. The transgressive generators $z_j \in H^*(n : \mathbb{F}_2)$, of Proposition 5.4 are primitive:

$$\mu^*(z_j) = z_j \otimes 1 + 1 \otimes z_j$$

from Borel [2, Proposition 20.1]. Let

$$\{\zeta_{i_1 \ldots i_p} \mid 3 \leq i_1 < \cdots < i_p \leq 2n,\ i_j \neq 2^l\}$$

be the linearly independent set in $H_*(n : \mathbb{F}_2) = H_*(\mathrm{Spin}(2n+1) : \mathbb{F}_2)$ dual to $z_{i_1} z_{i_2} \cdots z_{i_p} \in H^*(n : \mathbb{F}_2)$ under the Kronecker pairing. The primitivity of the z_j shows that if $\sum_{j=1}^{p} i_j \leq 2a_1 - 2$, then

$$\zeta_{i_1 \ldots i_p} = \zeta_{i_1} \vee \cdots \vee \zeta_{i_p},$$

where $\vee$ denotes the Pontryagin product and the right-hand side is independent of order.

Proposition 6.1. $\mu^*(z) = z \otimes 1 + 1 \otimes z + \sum z_i \otimes z_{2j}$ *in* $H^*(n : \mathbb{F}_2) \otimes H^*(n : \mathbb{F}_2)$, *where the sum runs over all pairs* (i, j) *such that* $i + 2j = 2a_1 - 1$. *In particular, this second sum is zero iff* $n = 3$ *or* $n = 2^\alpha$.

Proof. We will prove the results in three steps.

(a) The second factor in each summand of $\mu^*(z) - (z \otimes 1 + 1 \otimes z)$ must be a product of the z_{2j} (i.e. must lie in the Chow ring $A_n^*(\mathbb{F}_2) \subseteq H^*(n : \mathbb{F}_2)$).

(b) Each summand in the second sum must be of the form $z_i \otimes z_{2j}$.

(c) All pairs of indices (i, j) satisfying $i + 2j = 2a_1 - 1$, do occur in $\mu^*(z) - (z \otimes 1 + 1 \otimes z)$.

Step (a) follows from a diagram chase: put $G = \mathrm{Spin}(2n+1)$. Then we have

$$\begin{array}{ccc} G \times G & \xrightarrow{\mu} & G \\ {\scriptstyle 1\times\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ G \times X_n' & \xrightarrow{a} & X_n' \end{array}$$

where α is the left action of G on X_n'. Since $z = \pi^*(x)$ it follows that $\alpha^*(x)$ determines $\mu^*(z)$. Now

$$\alpha^*(x) = (z \otimes 1 + 1 \otimes x) + \sum \ldots$$

and the second sum has its second factor in $H^*(X_n' : \mathbb{F}_2)$ in degrees $\leq 2a_1 - 1$, hence in $A_n^*(\mathbb{F}_2)$ (see Corollary 3.5.1). This proves (a). For (b) consider the

Kronecker pairing

$$\langle \mu^*(z), \zeta_{i_1 \dots i_p} \otimes \zeta_{j_1 \dots j_q} \rangle \tag{6.1.1}$$

Now clearly $(\sum_{r=1}^{p} i_r) + (\sum_{s=1}^{q} j_s) = 2a_1 - 1$ for (6.1.1) to be nonzero and as each i_r, j_s is at least 3, each sum is $\leq 2a_1 - 2$. Thus

$$\zeta_{i_1 \dots i_p} = \zeta_{i_1} \vee \cdots \vee \zeta_{i_p}, \qquad \zeta_{j_1 \dots j_q} = \zeta_{j_1} \vee \cdots \vee \zeta_{j_q}$$

the products being independent of order. From (a) each j_s must be even, and so at least one i_r is odd; by the commutativity we can assume $i_r = i_p$. Now suppose that $p \geq 2$ or $q \geq 2$. Then $i_p + j_1 \leq 2a_1 - 2$, and so (6.1.1) is equal to

$$\begin{aligned} &\langle z, \zeta_{i_1} \vee \cdots \vee \zeta_{i_{p-1}} \vee \zeta_{j_1} \vee \zeta_{i_p} \vee \zeta_{j_2} \vee \cdots \vee \zeta_{j_q} \rangle \\ &\quad = \langle \mu^*(z), \zeta_{i_1 \cdots i_{p-1}} \zeta_{j_1} \otimes \zeta_{i_p j_2 \cdots j_q} \rangle \end{aligned}$$

which is zero because the second factor has an odd index. Thus $p = 1$ and $q = 1$ establishing (b).

At this point it follows that if $n = 2^\alpha$ or $n = 3$, then for numerical reasons there are no permissible indices i, j such that $i + 2j = 2a_1 - 1$; so $\mu^*(z) = z \otimes 1 + 1 \otimes z$.

To prove (c) fix a power of 2, and write $n = 2^\alpha + l$, $0 \leq l \leq 2^\alpha - 1$. In this range the element $z \in H^*(n+1 : \mathbb{F}_2)$ restricts to z in $H^*(n : \mathbb{F}_2)$ (comment following Lemma 5.4.3), and also the mod 2 spectral sequence of

$$\mathrm{Spin}(2n+1) \to \mathrm{Spin}(2n+3) \to V(2, 2n+3)$$

collapses, as one checks from Proposition 5.4(a) and 1.5. Hence we can use induction on l, starting with $l = 0$ for which the result was established in the previous paragraph; and using the inductive hypothesis for l, it is only necessary to prove that in

$$\mu^*(z) - (z \otimes 1 + 1 \otimes z) = \sum z_i \otimes z_{2j}$$

the elements z_{2l+1}, z_{2l+2} occur on the right-hand side. Since $l + 1 \geq 1$ and $\mathrm{Sq}^1(z) \neq 0$ when $n = 2^\alpha + l + 1$, this last assertion is verified by computing each side of the identity

$$\mu^*(\mathrm{Sq}^1(z)) = \mathrm{Sq}^1(\mu^*(z))$$

and comparing the results. $\square$

Proposition 6.2. *The analogue of Proposition* 6.1 *holds for* $H^*(\mathrm{Spin}(2n) : \mathbb{F}_2)$ *and* $\mu^*(z) = z \otimes 1 + 1 \otimes z$ *iff* $n \leq 4$.

Proof. If $n \neq 2^\alpha$, the mod 2 spectral sequence of

$$\mathrm{Spin}(2n) \to \mathrm{Spin}(2n+1) \to S^{2n}$$

degenerates at E_2, as one verifies from Proposition 5.4, the remarks following Lemma 5.4.3 and 1.4. Hence $\mu^*(z)$ in Spin$(2n)$ can be calculated from Proposition 6.1 by suppressing the summand involving z_{2n}, if any; and for $n \geq 5$ there will always be a term in $\mu^*(z) - (z \otimes 1 + 1 \otimes z)$ involving z_3 which will persist in Spin$(2n)$.

If $n = 2^\alpha$, then the bundle

$$\omega_{n-1} : \mathrm{Spin}(2n-1) \to \mathrm{Spin}(2n) \to S^{2n-1}$$

of Section 2 degenerates at E_2 in any case (Proposition 2.2), and the kernel of the restriction map $H^*(\mathrm{Spin}(2n) : \mathbb{F}_2) \to H^*((n-1) : \mathbb{F}_2)$ is the principal ideal generated by z_{2n-1}. Since $n = 2^\alpha$, $a_1 = 2n$ and so from Proposition 5.4 we see that in $\mu^*(z) - (1 \otimes z + z \otimes 1)$ there can be no summand involving z_{2n-1}; thus the formula for this difference can be read off from the restriction of $\mu^*(z) - (1 \otimes z + z \otimes 1)$ to $H^*(n-1 : \mathbb{F}_2)$. $\square$

Let $\zeta \in H_*(\mathrm{Spin}(k) : \mathbb{F}_2)$ be the class Kronecker dual to $z \in H^*(\mathrm{Spin}(k) : \mathbb{F}_2)$. Combining Propositions 6.1 and 6.2 we have the following proposition:

Proposition 6.3. *$\{\zeta_i, \zeta \mid 3 \leq i \leq k-1,\ i \neq 2^j\}$ is a simple set of generators for the Pontryagin ring $H_*(\mathrm{Spin}(k) : \mathbb{F}_2)$. Each generator has square zero, and the only nonzero commutators among the generators are*

$$[\zeta_i, \zeta_{2j}] = \zeta\,, \quad i + 2j = 2a_1 - 1\,.$$

Hence the ring is commutative for $k \leq 9$ or $k = 2^\alpha + 1$. $\square$

Notice that ζ is a central element in $H_*(\mathrm{Spin}(k) : \mathbb{F}_2)$, and $\{\zeta_{\mathrm{ev}}\}$, $\{\zeta_{\mathrm{odd}}\}$ generate exterior subalgebras.

We can now tackle the integral Pontryagin product. Define $\xi_i \in H_{4i-1}(n : \mathbb{Z})$ to be the Poincaré-dual of $\prod_{j \neq i} x_j \in H^*(n : \mathbb{Z})$.

Proposition 6.4. *The set $\{1, \xi_{i_1} \vee \cdots \vee \xi_{i_p} \mid 1 \leq i_1 < \cdots < i_p \leq n\}$ spans the free summand of $H_*(n : \mathbb{Z})$: it forms an exterior algebra iff $n = 3$ or $n = 2^\alpha$.*

Proof. The given set spans the algebra $H_*(n : \mathbb{Z}(\frac{1}{2}))$ and hence it is linearly independent over $\mathbb{Z}$, its $\mathbb{Z}$-span having a cokernel of two-power order in the free summand. Hence it suffices to show that the set is linearly independent over $\mathbb{F}_2$.

Using (5.4.6) and Proposition 6.3 one sees easily that

$$\bar{\xi}_j = \zeta_{4j-1} \quad \text{if } j \neq 2^i < s\,, \qquad \bar{\xi}_s = \zeta\,,$$
$$\bar{\xi}_j = \zeta_{2j-1}\zeta_{2j} \quad \text{if } j \neq 2^i\,, \tag{6.4.1}$$

whence $\prod_1^n \bar{\xi}_i$ = fundamental class in $H_*(n : \mathbb{F}_2)$ proving the linear independence (as in Proposition 4.5).

Next, $\xi_i \vee \xi_i$ is a 2-torsion element for reasons of degree, so $\xi_i \vee \xi_i = \bar{\xi}_i \vee \bar{\xi}_i = 0$ as one sees form the formulae above. The commutator ideal of $H_*(n : \mathbb{Z})$ is in any case within the torsion ideal, so $[\xi_k, \xi_j] = [\bar{\xi}_k, \bar{\xi}_j]$ and from Proposition 6.3 one finds that if $j, k \neq 2^i$ and $j + k = a_1$, then $[\bar{\xi}_k, \bar{\xi}_j] = \zeta \vee (\zeta_{2k-1} \vee \zeta_{2j} + \zeta_{2j-1} \vee \zeta_{2k})$ if $j \neq 2^i$, $k = 2^1$ and $2k + j = a_1$, then $[\bar{\xi}_k, \bar{\xi}_j] = \zeta \vee (\zeta_{2j-1})$ and the other commutators of the elements $\bar{\xi}_i$ are zero. □

Remarks. (a) The value of the nonzero commutators above can of course be expressed in terms of integral elements, as we shall show below (Proposition 6.6) once we introduce the torsion generators of $H_*(n : \mathbb{Z})$.

(b) The analogue of Proposition 6.4 is true for $H_*(\mathrm{Spin}(2n+2) : \mathbb{Z})$ with the ξ_i defined as Poincaré-duals of $(\prod_{j \neq i} x_j)\varepsilon$ and η the dual of $\prod_1^n x_i$. In this case the free summand is not an exterior algebra for $n \geq 4$ by Proposition 6.3 and 6.4 (see (6.4.2)).

(c) Henceforth we drop the symbol '$\vee$' for the Pontryagin product. To define the torsion generators in homology, we use the universal-coefficient sequence

$$0 \to H_*(\mathrm{Spin}(k) : \mathbb{Z}) \otimes \mathbb{F}_2 \to H_*(\mathrm{Spin}(k) : \mathbb{F}_2)$$
$$\xrightarrow{\lambda} \mathrm{Tor}(H_{*-1}(\mathrm{Spin}(k) : \mathbb{Z}), \mathbb{F}_2) \to 0$$

identifying the last term with the torsion subgroup of $H_*(\mathrm{Spin}(k) : \mathbb{Z})$ (which is of exponent 2 by dualizing the result in cohomology). The composite of λ followed by reduction mod 2

$$\bar{\lambda} : H_*(\mathrm{Spin}(k) : \mathbb{F}_2) \to H_{*-1}(\mathrm{Spin}(k) : \mathbb{F}_2) \tag{6.4.2}$$

is the homology operation dual to Sq^1 and it is a derivation of the Pontryagin product; hence $\bar{\lambda}$ is completely specified by

$$\bar{\lambda}(\zeta_{2i}) = \zeta_{2i-1}\,, \qquad \bar{\lambda}(\zeta_{2i-1}) = 0\,, \qquad \bar{\lambda}(\zeta) = 0\,,$$

the last equation following most easily from $\bar{\xi}_s = \zeta$ (see (6.4.1)). Since the subalgebra of $H_*(\mathrm{Spin}(k) : \mathbb{F}_2)$ generated by $\{\zeta_{2i}\}$ is commutative, for $I \subseteq \{i \mid 3 \leq i \leq k - 1/2,\ i \neq 2^j\}$ the product $\zeta_{2I} = \prod_I \zeta_{2i}$ is well defined (independent of the ordering in I) and we set

$$\mu_I = \bar{\lambda}(\zeta_{2I}) \in H_{q-1}(\mathrm{Spin}(k) : \mathbb{F}_2)\,, \quad \text{where } q = 2\sum_I i\,.$$

Here if E is the empty set, then $\zeta_E = 1$ so $\mu_E = 0$. Putting $I = \{i_1, \ldots, i_p\}$ the remarks above on $\bar{\lambda}$ give $\mu_I = \bar{\lambda}(\prod_I \zeta_{2i}) = \sum_{r=1}^{p} \zeta_{2i_1} \cdots \zeta_{2i_r - 1} \cdots \zeta_{2i_p}$ in $H_{q-1}(\mathrm{Spin}(k) : \mathbb{F}_2)$. When I is a singleton $\{i\}$ we denote μ_I by μ_i.

Proposition 6.5. *The set $\{1, \xi_1, \ldots, \xi_n, \mu_I\}$ generates the Pontryagin ring $H_*(n : \mathbb{Z})$. For $H_*(\mathrm{Spin}(2n+2) : \mathbb{Z})$ we simply add η to this set.*

Proof. We already know that $\{1, \xi_1, \ldots, \xi_n\}$ generates the free part of $H_*(n : \mathbb{Z})$ and the torsion subgroup is mapped onto by $H_*(n : \mathbb{F}_2)$ under λ. Hence from Proposition 6.3 it suffices to show that for all permissible ordered index sets $I = \{i_1, \ldots, i_p\}$, $\lambda(\zeta_I)$ and $\lambda(\zeta_I \zeta)$ are in the subring generated by $\{1, \xi_1, \ldots, \xi_n, \mu_J\}$. The second case reduces to the first because $\lambda(\zeta_I \zeta) = \lambda(\zeta_I)\bar{\xi}_s$.

To write out $\lambda(\zeta_I)$ as an element of the subring in $\{\xi_j, \mu_I\}$ we use induction on $p = |I|$, the case $p = 1$ being clear. If I does not contain a pair of indices summing to $2a_1 - 1$, then $\zeta_I = \prod \zeta_{\mathrm{odd}} \prod \zeta_{\mathrm{even}}$ and so

$$\bar{\lambda}(\zeta_I) = \prod \zeta_{\mathrm{odd}} \bar{\lambda}\left(\prod \zeta_{\mathrm{ev}}\right) = \prod \zeta_{\mathrm{odd}} \mu_J ,$$

J being the set of $\frac{1}{2}$ times the even elements in I. Now among the ζ_{2i-1} appearing above, if $i = 2^k$, then $\zeta_{2i-1} = \bar{\xi}_{2^k - 1}$, otherwise $\zeta_{2i-1} = \mu_i$.

It remains to consider the case in which there is at least one pair of indices i_r, i_s in I with $i_r + i_s = 2a_1 - 1$. Again we try to effect the rearrangement above, except that for each i_r, i_s we may introduce another monomial $\zeta_{I'} \zeta$ where $I' = I - \{i_r, i_s\}$. Hence by the induction hypothesis and the reduction above, we are done. $\square$

Note that we have proved something more; namely that the torsion subgroup of $H_*(\mathrm{Spin}(k) : \mathbb{Z})$ is generated by $\{\xi_{2^i}, \mu_J\}$. An immediate consequence of the discussion above is the following proposition:

Proposition 6.6. *In $H_*(n : \mathbb{Z})$ we have*

(a) *if $j, k \neq 2^i$ and $j + k = a_1$, then $[\xi_j, \xi_k] = \xi_s \mu_J$, where $J = \{j, k\}$,*

(b) *if $j \neq 2^i$, $k = 2^i$ and $j + 2k = a_1$, then $[\xi_j, \xi_k] = \xi_s \mu_k$.*

In $H_*(\mathrm{Spin}(2n+2) : \mathbb{Z})$ we have

(c) *η commutes with ξ_j, $j = 2^i \leq n$,*

(d) *if $k = a_1 - (n+1) \neq 2^i$, then $[\xi_k, \eta] = \xi_s \mu_k$.* $\square$

To describe the product relations among the $\{\xi_j, \mu_I\}$ we define a permissible index set to be *special* iff it is a pair of the form $\{i, a_1 - i\}$.

Proposition 6.7. (1) *If $i \in I$, $\xi_i \mu_I = \mu_I \xi_i = 0$.*

(2) *If $|I \cap J| \geq 1$, then $\mu_I \mu_J = 0$* except *in the following cases. Put $L = I \cup J$, $M = L - (I \cap J)$.*

(a) *If* $I \cap J$ *is special,*

$$\mu_I \mu_J = \xi_s \mu_M .$$

(b) *If* $I \cap J = \{k\}$, *then*

$$\mu_I \mu_J = \begin{cases} \xi_k \mu_M = \mu_k \mu_L , & \text{if } a_1 - k \in J , \\ \mu_M \xi_k = \mu_L \mu_k , & \text{if } a_1 - k \in I \end{cases}$$

(*and the expressions are equal if* $a_1 - k \not\in L$).

(3) *If* $I \cap J = \emptyset$ *and* $I = \{i_1, \ldots, i_p\}$, *then*

$$\mu_I \mu_J = \sum_{r=1}^{p} \mu_{i_r} \mu_{L_r}, \; L_r = I \cup J - \{i_r\}$$

provided I does not contain a special set; *and*

$$\mu_I \mu_J = \sum_{1}^{p} \mu_{i_r} \mu_{L_r} + \xi_s \mu_{I'}$$

where $I' = I - ($*given special set*$)$.

Proof. All the verifications are carried out by reduction mod 2: (1) follows from $\bar{\xi}_1 = \zeta_{2i-1}\zeta_{2i} = \zeta_{2i}\zeta_{2i}$ and $\bar{\mu}_I = \bar{\lambda}(\zeta_{2i}\zeta_{2I'}) = \bar{\lambda}(\zeta_{2I'}\zeta_{2i})$ where $I' = I - \{i\}$. For (2) and (3) it is simpler to develop an analogous formula for $\bar{\mu}_I \zeta_{2J}$ and then apply $\bar{\lambda}$. For example, for the first case (2)(b), write $\mu_I = \lambda(\zeta_{2I'}\zeta_{2k})$, $\zeta_{2J} = \zeta_{2k}\zeta_{2J'}$ ($I' = I - \{k\}$, $J' = J - \{k\}$) expand and multiply out: the hypothesis implies that $[\zeta_{2I'}, \zeta_{2k-1}] = 0$ giving $\bar{\mu}_I \zeta_{2J} = \bar{\xi}_k \xi_{2M} = \mu_k \bar{\mu}_L$.

(3) is proved by induction on $p = |I|$ to give $\bar{\mu}_I \zeta_{2j} = \sum_1^p \mu_{i_r} \zeta_{2L_r}$. □

Note that the second set of equations in (2)(b) above give $\xi_k \mu_J$ and $\mu_J \xi_k$ when $k \neq 2^i$ and $k \not\in J$.

Given Proposition 6.6, we fix permissible index sets I, J to describe the remaining commutators and we put $L = \{i \in I \mid a_1 - i \in J\}$.

Proposition 6.8. (1) *If* $k = 2^i < n$ *and* $a_1 = 2k \in J$, *then*

$$[\xi_k, \mu_J] = \xi_s \mu_{J'}, \quad \text{where } J' = J - \{a_1 - 2k\}.$$

(2) $[\mu_I, \mu_J] = 0$ *unless* $|I \cap J| \leq 1$ *and*

(a) *If* $I \cap J = \{k\}$ *and* $a_1 - k \in I \cup J$,

$$[\mu_I, \mu_J] = \xi_s \mu_k , \quad \text{where } K = I \cup J - \{a_1 - k\}$$

(b) *If* $I \cap J = \emptyset$, $L \neq \emptyset$, *then*

$$[\mu_I, \mu_J] = \xi_s \sum \mu_{H_l} H_l = I \cup J\{-l, a_1 - l\} .$$

Proof. As always, the verifications proceed by reduction mod 2, and (1) is direct. For (2) it is easier to follow the method of proof in Proposition 6.7 and obtain a formula for $[\mu_I, \zeta_{2J}]$, and then apply $\bar{\lambda}$.

(2)(a) can be deduced from Proposition 6.7(2)(b). For (2)(b) we fix an $l \in L$ and observe

$$[\mu_I \zeta_{2J}] = [\mu_{I'}, \zeta_{2M}] + \zeta\zeta_{2I'}\zeta_{2J'},$$

where $M = J \cup \{l\}$ and $I' = I - \{l\}$, $J' = J - \{a_1 - l\}$. Then induction on $|L|$ gives the desired formula. □

Note that when $k \neq 2^i$, $[\xi_k, \mu_J]$ is subsumed by Proposition 6.7(1), 6.7(2)(b) and 6.8(2)(a).

Theorem 3. *$H_*(\mathrm{Spin}(k) : \mathbb{Z})$ is generated by $\{1, \xi_i, \mu_J\}$ subject only to Propositions 6.6, 6.7 and 6.8 (and $2\mu_J = 0$).*

Proof. Let $H_*(k)$ be the (noncommutative) graded, connected algebra over $\mathbb{Z}$ generated by $\{1, \xi_i, \mu_J\}$ subject to the given relations. By Propositions 6.5 to 6.8 there is a surjection $\varphi : H_*(k) \to H_*(\mathrm{Spin}(k) : \mathbb{Z})$; and by Proposition 6.4 φ is an isomorphism of the free parts. We must show that φ is an isomorphism on the torsion ideals also.

We will first do the argument for $k = 2^{\alpha+1} + 1$, when these algebras are commutative; and then indicate the minor modifications necessitated by the existence of nonzero commutators. In the case at hand, the argument in Proposition 6.5 shows that the torsion ideal of $H_*(2^\alpha : \mathbb{Z})$ is a free module over $\Lambda^*(\xi_1, \xi_2, \ldots, \xi_s)$ generated by monomials $\mu_{i_1} \cdots \mu_{i_p}$ and $\mu_{i_1} \cdots \mu_{i_p}\mu_J$, $|J| \geq 2$. Hence it suffices to show that the same is true in $H_*(k)$; that there is a *canonical* reduction of every torsion element to a $\Lambda^*(\xi_1, \ldots, \xi_s)$-combination of monomials $\mu_{i_1} \cdots \mu_{i_p}\mu_J$ and finally that every relation which holds among such monomials in $H_*(2^\alpha : \mathbb{Z})$ holds also in $H_*(k)$, $k = 2^{\alpha+1} + 1$.

So consider a torsion element $\xi_{k_1} \cdots \xi_{k_p}\mu_{J_1} \cdots \mu_{J_q}$ in $H_*(k)$. We can assume that no k_r is a power of 2; then Proposition 6.7(1) gives zero if some k_r lies in some J_s: and if not, then Proposition 6.7(2)(b) gives $\xi_j\mu_M = \mu_j\mu_L$, where $L = M \cup \{j\}$. Thus we have a canonical reduction to the form $\mu_{i_1} \cdots \mu_{i_t}$, $\mu_{K_1} \cdots \mu_{K_u}$. If $m \geq 1$, then Propositions 6.7(2)(b) and (3) give a reduction (by using the ordering given) to the desired form. So it remains to find a generating set of relations among $\mu_{i_1} \cdots \mu_{i_p}$ and $\mu_{i_1} \cdots \mu_{i_p}\mu_J$ in $H_*(2^\alpha : \mathbb{Z})$. There is an obvious one; for any K, setting $K_k = K - \{k\}$ we have $\sum_K \mu_k\mu_{K_k} = 0$ because $\sum_K \bar{\mu}_k \zeta_{2K_k} = \bar{\mu}_k$ and we apply $\bar{\lambda}$. This exists in $H_*(k)$ by Proposition 6.7(3); fix a $k_0 \in K$, put $I = \{k_0\} J = K - \{k_0\}$ and use $\mu_I\mu_J = \mu_J\mu_I$. A dimension count shows that these are a generating set. (Essentially the same proof works for $\mathrm{SO}(n)$, see 7.7.)

When $k-1 \neq 2^j$, so that the algebras are not commutative, the general torsion element is a product of monomials of the form $\xi_{i_1} \cdots \xi_{i_p} \mu_{J_1} \cdots \mu_{J_a}$. At the expense of introducing summands in the ideal (ξ_s) we move each ξ_i next to a μ_J if $i \in J$; and if not, then we move it until we can use Proposition 6.7(2)(b). Thereafter the reduction proceeds as above, and we end up with terms of the form $\mu_{k_1} \cdots \mu_{k_r}$ or $\mu_{k_1} \cdots \mu_{k_r} \mu_{J_1} \cdots \mu_{J_2}$ *plus* a term from (ξ_s). The relations among these are similar to those above except that

$$\bar{\mu} = \sum \bar{\mu}_n \zeta_{2K_k} + \xi_s(?)$$

and the rest of the argument is similar. □

7. The case of SO(k)

We begin by remarking that the double-covering along the fibres

$$\begin{array}{cccc}
\xi_n: & T \rightarrow & \mathrm{Spin}(k) \rightarrow & F(k) \\
& \downarrow p & \downarrow & \| \\
\xi_n': & T' \rightarrow & \mathrm{SO}(k) \rightarrow & F(k)
\end{array}$$

gives an isomorphism of spectral sequences and cohomology rings for any coefficient ring containing $\frac{1}{2}$. Now we shall run through the analogues of Sections 4 to 6 for $\mathrm{SO}(2n+1)$, indicating the required modifications for the even case where necessary. In particular, these assertions will show that Theorem 1 (Section 2) holds for SO as well.

7.1. $E_2(\xi_n' : R) = \Lambda_R^*(t_1, \ldots, t_n) \otimes H^*(F(2n+1) : R)$ and the analogue of Proposition 4.1 is immediate from the remarks above. The elements $\theta_1, \ldots, \theta_n$ of (4.2.1) are defined in $E_2(\xi_n' : \mathbb{Z})$, and by Propositions 4.3 and 4.5 they generate an exterior algebra $\mathscr{E}_n'$ which is additively a summand, isomorphic to the free part of $E_3(\xi_n' : \mathbb{Z})$. For $\mathrm{SO}(2n+2)$ the element τ_{n+1} has to be added to $\{\theta_1, \ldots, \theta_n\}$.

7.2. From Corollary 3.5.1 and Proposition 3.7

$$E_2(\xi_n' : \mathbb{F}_2) = \Lambda^*(t_1, \ldots, t_n) \otimes H^*(U(n)/T' : \mathbb{F}_2) \otimes A_n'(\mathbb{F}_2)$$

and as a cochain complex under d_2 it splits as a product of $A_n'(\mathbb{F}_2)$, $d_2 = 0$ and $E_3(\xi_n' : \mathbb{Z})$. For $\mathrm{SO}(2n+2)$ the element τ_{n+1} has to be added to $\{\theta_1, \ldots, \theta_n\}$.

$$E_3(\xi_n' : \mathbb{F}_2) = \Lambda^*(\tau_1, \ldots, \tau_n) \otimes A_n'(\mathbb{F}_2)$$

and this is E_∞ as well since the fibre-degrees of the generators are ≤ 1. The mod 2 reductions of the θ_j are given by

$$\begin{aligned} \bar{\theta}_j &= \tau_{2j} + \tau_j \bar{\gamma}_j\,, && \text{if } 1 \leq j \leq n/2\,, \\ \bar{\theta}_j &= \tau_j \bar{\gamma}_j\,, && \text{if } n/2 < j \leq n\,. \end{aligned} \tag{7.2.1}$$

7.3. Let T_n' = torsion subgroup of $E_3(\xi_n' : \mathbb{Z})$. From 7.1, T_n' is a group of 2-power order and by the argument of Proposition 4.7, $r_2(T_n') = 2^{n-1}(2^n - 1)$. The formula for the Bockstein map $\bar{\delta} : E_3(\xi_n' : \mathbb{F}_2) \to E_3(\xi_n' : \mathbb{F}_2)$ is $\bar{\delta}(\tau_k) = \bar{\gamma}_k$, $\bar{\delta}(\bar{\gamma}_k) = 0$. Here the paradigm example of Lemma 4.8.1 yields $H^*(E_3, \bar{\delta}) = \mathscr{E}_n' \otimes \mathbb{F}_2$ which implies $2 \cdot T_n' = 0$. Then $E_3(\xi_n' : \mathbb{Z}) = E_\infty(\xi_n' : \mathbb{Z})$ as in Propositions 4.9 answering affirmatively the question of Kač [4] for $G = \mathrm{SO}(k)$. If we define B_n' to be the $\mathbb{Z}$-module in $E_\infty(\xi_n' : \mathbb{Z})$ generated by 1 and

$$\beta_I' = \delta\left(\prod_I \tau_{2i-1}\right), \quad I \subseteq \{1, 2, \ldots, m\}\,, \ |I| \geq 2\,,$$

then $E_\infty(\xi_n' : \mathbb{Z})$ is generated by $\{\theta_i, \gamma_j, \beta_K'\}$ as in Proposition 4.11. Note that in this case there is a β_I' as soon as $m \geq 2$—i.e. $n \geq 3$. The analogues of Propositions 4.13 to 4.17 all go through; we will not list the relations as they are only intermediate steps in proving the corresponding assertions for $H^*(\mathrm{SO}(2n+1) : \mathbb{Z})$ (see 7.5).

We now proceed to the parallels with Section 5.

7.4. Lemma 5.2 holds for $E_\infty(\xi_n' : R)$ and so we can define $x_i' \in H^{4j-1}(\mathrm{SO}(2n+1) : \mathbb{Z})$ as the image of $\theta_j \in E_\infty^{4i-2,1}(\xi_n' : \mathbb{Z})$, $u_j \in H^{2i}(\mathrm{SO}(2n+1) : \mathbb{Z})$ as the image of $\gamma_i \in E_\infty^{2i,0}(\xi_n' : \mathbb{Z}) = A_n'^{2i}(\mathbb{Z})$. (For $\mathrm{SO}(2n+2)$ we add the element ε' corresponding to τ_{n+1}.) The square-free products of $\{1, x_i'\}$ form a basis for the free part of $H^*(\mathrm{SO}(2n+1) : \mathbb{Z})$ and under the covering map p

$$\begin{aligned} p^*(x_i') &= x_i\,, && \text{if } i \text{ is not a power of } 2\,, \\ p^*(x_j') &= x_j\,, && \text{modulo } H^*(X_n : \mathbb{Z})\text{-linear combinations} \\ & && \text{of } x_{2^i}, \text{ if } 2^i < j = 2^l < s\,, \\ p^*(x_s') &= 2x_s\,, && \text{modulo lower order terms} \end{aligned}$$

(and $p^*(\varepsilon') = \varepsilon$ in the even case).

Using Lemma 5.4.2 and (7.2.1)

$$\begin{aligned} \bar{x}_j &= h_{4j-1} + h_{2j-1}h_{2j}\,, && \text{if } 1 \leq j \leq n/2\,, \\ \bar{x}_j' &= h_{2j-1}h_{2j}\,, && \text{if } n/2 < j \leq n \end{aligned} \tag{7.4.1}$$

and of course $\bar{u}_i = h_{2i}$. The Bockstein elements v'_I are uniquely (mod 2 reductions $\bar{v}'_I = \mathrm{Sq}^1(\prod_I h_{4i-3})$, where $I \subseteq \{1, 2, \dots, m\}$, from [3] and [7] the torsion exponent of $H^*(\mathrm{SO}(k) : \mathbb{Z})$ is 2.

7.5. The set $\{1, x'_i, u'_j, v'_K\}$ generates the ring $H^*(\mathrm{SO}(2n+1) : \mathbb{Z}$ for $H^*(\mathrm{SO}(2n+2) : \mathbb{Z})$ and a generating list of relations is given

(a) For $1 \le i \le (2n+1)/4$, $x_i'^2 = u'_{4i-1} + u'_{2i-1}u'_{2i}$.

(b) For $(2n+1)/4 < i \le n/2$, $x_i'^2 = u'_{2i-1}u'_{2i}$.

(c) For $n/2 < i \le n$, $x_i'^2 = 0$.

(d) For $1 \le j \le m$,

$$k_{rj} = 2^{\alpha_j} - \sum_0^i 2^r, \qquad l_{ij} = 2^i(2j-1), \qquad \sum x_{l_{ij}} u_{2j-1}^{k_{ij}} = ($$

Using notation analogous to Proposition 5.7

(e) $\sum_I u'_{2i-1} \otimes v'_I = 0$, $|I| \ge 3$.

(f) If $k \in I$, $\tilde{s}'_k \otimes v'_I = 0$.

(g) If $k \not\in I$, $s'_k \otimes v'_I + u_{2k-1}^{ak-1} \otimes v'_J = 0$, $j = I \cup \{k\}$.

(h) If $i \cap J = K \ne \emptyset$ and $I' = I - K$, $J' = J - K$,

$$v'_I v'_J = \Big(\prod_K u'_{4k-3}\Big) v'_{I'} v'_{J'} + \Big(\prod_K u'_{2k-1}\Big) v'_{I \cup J}\,.$$

(i) If $I \cap J = \emptyset$, $I = \{i_1, \dots, i_p\}$ and $L_r = I \cup J - \{i_r\}$, $1 \le r \le$

$$v'_I v'_J = \sum_1^p u'_{2i-1} \otimes v'_{L_r}\,.$$

There are no relations involving ε' (except $\varepsilon'^2 = 0$), confirming Theorem 1 for SO.

Adding to these relations in the Chow ring $u_i'^2 = u'_{2i}$ iff $1 \le$ $n < i \le 2n$ (or $2n+1$) and $\varepsilon'^2 = 0$ in the even case gives the rings for $k = 2n+1$, $2n+2$.

We now move on to describe the Pontryagin product in $H_*($S will of course be commutative, as one easily deduces from t $H_*(\mathrm{SO}(k) : \mathbb{F}_2)$ and $H_*(\mathrm{SO}(k) : \mathbb{Z}(\tfrac{1}{2}))$.

7.6. From Borel [3] we know that the Pontryagin ring H_* $\Lambda^*(\zeta'_1, \dots, \zeta'_{k-1})$, where ζ'_1 is Kronecker dual to h_i. From Le Proposition 6.3 $p_*(\zeta_i) = \zeta'_i$, $p^*(\zeta) = 0$.

Define $\xi'_i \in H_{4i-1}(\mathrm{SO}(2n+1) : \mathbb{Z})$ to be the Poincaré dual t analogous definitions involving ε' in the even case); then from $\bar{\xi}'_i = \zeta'_{2i-1}\zeta'_{2i}$, $1 \le j \le n$, and it follows along the lines of Prop $\{1, \xi'_i\}$ generate the exterior subalgebra of $H_*(\mathrm{SO}(2n+1) : \mathbb{Z})$ v free summand.

In the notation of (6.4.2) we define torsion elements by $\mu'_I = \lambda(\prod_I \zeta'_{2i}) \in H_{q-1}(\mathrm{SO}(2n+1):\mathbb{Z})$, where $q = 2\sum_I i$ and $I \subseteq \{1,2,\ldots,n\}$. Arguing as in Proposition 6.5, the set $\{1, \xi'_i, \mu'_I\}$ generates $H_*(\mathrm{SO}(2n+1):\mathbb{Z})$ (with the dual η' of ε' added in the even case) but in contrast with the Spinor case, $\{\mu'_I\}$ suffices to generate the torsion ideal.

7.7. The structure of $H^*(\mathrm{SO}(k):\mathbb{Z})$ is now given as follows.

Proposition. *The obvious map from $\Lambda^*(\xi'_1,\ldots,\xi'_n) \otimes \Lambda^*(\mu'_I)$ to $H_*(\mathrm{SO}(2n+1):\mathbb{Z})$ is onto with kernel generated by*

(a) $2\mu'_I$,

(b) *if* $i \in I$, $\xi'_i \mu'_I$,

(c) *if* $|I \cap J| \geq 2$, $\mu'_I \mu'_J$,

(d) *if* $I \cap J = \{k\}$, $L = I \cup J$, $M = L - \{k\}$,

$$\mu'_I \mu'_J = \xi'_k \mu'_M \quad \text{and} \quad \mu'_I \mu'_J = \mu'_k \mu'_L ,$$

(e) *if* $I \cap J = \emptyset$, $I = \{i_1,\ldots,i_p\}$ *and* $L_r = I \cup J - \{i_r\}$,

$$\mu'_I \mu'_J = \sum_{r=1}^{p} \mu'_{i_r} \mu'_{L_r} .$$

(In the even case we add ε' to the generators ξ'_i.) □

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References

[1] B. Boe and H. Hiller, A Pieri formula for $\mathrm{SO}(2n+1)/U(n)$ and $\mathrm{Sp}(n)/U(n)$, Adv. in Math. 62 (1) (1986) 49–67.

[2] A. Borel, Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts, Ann. of Math. 57 (1953) 115–207.

[3] A. Borel, Sur l'homologie et la cohomologie des groupes de Lie compacts connexes, Amer. J. Math. 76 (1954) 273–342.

[4] V. Kač, Torsion in cohomology of compact Lie groups and Chow rings of reductive algebraic groups, Invent. Math. 80 (1985) 69–79.

[5] J. Kojima, On the Pontryagin product mod 2 of Spinor groups, Memoirs Fac. Sci. Kyushu Univ. Ser. A 11 (1957) 1–14.

[6] R. Marlin, Les anneaux de Chow de SO(n), Spin(n), G_2 et F_4, Thesis Université de Paris, Orsay, 1974.

[7] C.E. Miller, The topology of rotation groups, Ann. of Math. 57 (1953) 90–114.
[8] A.S. Svarč, Cohomology of Spinor groups, Doklady Akad. Nauk USSR 104 (1955) 26–29 (in Russian).
[9] J. Tate, The homology of Noetherian rings and local rings, Illinois J. Math. 1 (1957) 14–27.
[10] H. Toda and K. Watanabe, The integral cohomology rings of F_4/T and E_6/T, J. Math. Kyoto Univ. 14 (1974) 257–286.